Continuous K-theory and Cohomology of Nonarchimedean Analytic Spaces

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Motivation

Let k be a complete discretely valued field with uniformiser π. Let X be a Berkovich space of dimension d.

- Assuming the existence of a nice model $\mathcal{X}$, one can compute the cohomology groups $\mathrm{H}^*(X;\mathbb{Z})$ via the Berkovich skeleton by the intersection complex $\Delta(\mathcal{X}/\pi)$ of the special fibre $\mathcal{X}/\pi$.
- Assuming resolution of singularities over k and the model $\mathcal{X}$ to be regular, then $\Delta(\mathcal{X}/\pi)$ also computes the continuous K-theory group $\mathrm{K}^{\mathrm{cont}}_{-d}(X)$.

Thus one might ask whether $\mathrm{K}^{\mathrm{cont}}_{-d}(X) \cong \mathrm{H}^d(X;\mathbb{Z})$ also holds without any assumptions on the model and without resolution of singularities.

Continuous K-theory

Let A_0 be a complete π-adic ring for some $\pi \in A_0$. Its continuous K-theory is the pro-spectrum

$$\mathrm{K}^{\mathrm{cont}}(A_0) := \text{“}\lim_n\text{”}\ \mathrm{K}(A_0/\pi^n)$$

where K is the nonconnective algebraic K-theory spectrum. Setting $A := A_0[\pi^{-1}]$, we define its continuous K-theory as the pushout

$$\begin{array}{ccc} \mathrm{K}(A_0) & \longrightarrow & \mathrm{K}(A) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(A_0) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A). \end{array}$$

in the ∞-category of pro-spectra. If $A = A_0'[\lambda^{-1}]$ for another complete λ-adic ring A_0', one obtains a weakly equivalent pro-spectrum, i.e. the pro-homotopy groups are pro-isomorphic. This notion was suggested by Morrow [3] and studied by Kerz-Saito-Tamme [1].

Zariski-Riemann spaces

Geometrically, $\mathrm{Spec}(A)$ is an open subset of $\mathrm{Spec}(A_0)$ whose closed complement is $\mathrm{Spec}(A_0/\pi)$. An admissible model of A is a reduced scheme $\mathcal{X}$ together with a projective map $\mathcal{X} \to \mathrm{Spec}(A_0)$ which is an isomorphism over $\mathrm{Spec}(A)$.

Without assuming the existence of a nice model of A one can alternatively study all models at once.

Thus we work with the Zariski-Riemann space $\mathcal{Z}(A)$ which is the cofiltered limit of all admissible models of A within the category of locally ringed spaces. Its special fibre $\mathcal{Z}(A)/\pi$ is isomorphic to Huber's adic spectrum $\mathrm{Spa}(A, A_0)$.

Main Result

Theorem 1 (D.). Let A be an affinoid k-algebra of dimension d. Then there is an isomorphism

$$\mathrm{K}^{\mathrm{cont}}_{-d}(A) \xrightarrow{\cong} \mathrm{H}^d(\mathcal{M}(A);\mathbb{Z})$$

where $\mathcal{M}(A)$ is the Berkovich spectrum of A.

Sketch of proof:

- $\mathrm{K}^{\mathrm{cont}}_{-d}(A) \cong \mathrm{K}^{\mathrm{cont}}_{-d}(\mathcal{Z}(A))$ using pro-cdh descent and platification par éclatement.
- $\mathrm{K}^{\mathrm{cont}}_{-d}(\mathcal{Z}(A)) \cong \mathrm{K}_{-d}(\mathcal{Z}(A)/\pi)$ via the Zariski descent spectral sequence. In particular, it is a constant pro-object.
- $K_{-d}(\mathcal{Z}(A)/\pi) \cong \mathrm{H}^d_{\mathrm{cdh}}(\mathcal{Z}(A)/\pi;\mathbb{Z})$ by using [2, Thm. D].
- $\mathrm{H}^*_{\mathrm{cdh}}(\mathcal{Z}(A)/\pi;\mathbb{Z}) \cong \mathrm{H}^*_{\mathrm{Zar}}(\mathcal{Z}(A)/\pi;\mathbb{Z})$ by pulling back covers and using completeness.
- $\mathrm{H}^*_{\mathrm{Zar}}(\mathcal{Z}(A)/\pi;\mathbb{Z}) \cong \mathrm{H}^*(\mathcal{M}(A);\mathbb{Z})$ is classical rigid geometry [4].

Future Work

- Globalise Theorem 1 to arbitrary (nice) nonarchimedean analytic spaces.
- Study Zariski-Riemann spaces as almost regular models in order to avoid the assumption of resolution of singularities for other situations.

References

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