A COUNTEREXAMPLE TO PRO-CDH DESCENT FOR NON-NOETHERIAN SCHEMES

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Abstract. Pro-cdh descent of algebraic K-theory holds for finite-dimensional noetherian schemes due to a result of Kerz-Strunk-Tamme. We provide a counterexample showing the failure of pro-cdh-descent in the non-noetherian setting.

Let F be a field and let

$$A := F[y, x_i \mid i \in \mathbb{N}]/(yx_0, yx_{i+1} - x_i \mid i \geq 0).$$

We also write $x = x_0$ Hence in A we have

$$0 = yx, \quad x = y^i x_i, \quad x^2 = x(yx_1) = (xy)x_1 = 0$$

We put

$$I = (y), \quad J = (x),$$

and claim that

$$\begin{tikzcd} \mathrm{Spec}(A/I+J) \arrow[r] \arrow[d] & \mathrm{Spec}(A/J) \arrow[d] \\ \mathrm{Spec}(A/I) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

is a finitely presented abstract blow-up square. Indeed, all maps are finitely presented closed immersions. Also, as $yx = 0$, it follows that $J \otimes A[y^{-1}] = 0$, and hence the right-hand vertical map is an isomorphism outside $\mathrm{Spec}(A/I)$. Note moreover, that the left vertical map is also an isomorphism, as $J \subseteq I$ (in fact $x = y^n x_n \in I^n$ for every n).

Now assume, "pro-cdh descent" would hold for this square. Then the square of pro-spectra

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d] & \text{“}\lim\limits_n\text{”}\, K(A/I^n) \arrow[d] & \\ K(A/J) \arrow[r] & \text{“}\lim\limits_n\text{”}\, K(A/I^n + J) \arrow[r, equal] & \text{“}\lim\limits_n\text{”}\, K(A/I^n) \end{tikzcd}$$

would be (weakly) cartesian. As in this diagram, the right vertical map is clearly an equivalence, the left vertical map also had to be an equivalence. In particular, we would get an isomorphism

$$K_1(A) \xrightarrow{\cong} K_1(A/J).$$

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As the units are a direct factor on both sides, this would imply an isomorphism

$$A^\times \xrightarrow{\cong} (A/J)^\times.$$

But as $x^2 = 0$, we have that $1+x \in A^\times$ is a non-trivial element in the kernel. Contradiction.