ALGEBRA AND TOPOLOGY, PART II

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ABSTRACT. This is lecture notes for the second half of a course I am teaching together with Lorenzo Mantovani and Emil Jacobsen at University of Zurich during the spring term 2020. These notes cover the content of the lectures. Here and there, some additional remarks and details are added.

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CONTENTS

Prerequisites: The reader should be familiar with the content of chapters 1,2, and 5 in Shapira's note [AT] which were treated in the first half of the course.

Caveat: These notes are work in progress, so you should read them with caution! If you notice any mistakes or typos, please send me an email.

1. Motivation for cohomology

On the complex numbers $\mathbf{C}$ there is the exponential function

$$\exp\colon \mathbf{C} \longrightarrow \mathbf{C}^{\times}, \quad x \mapsto \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

More generally, the exponential function gives rise to map of abelian sheaves

$$\exp\colon \mathcal{O}_{\mathbf{C}} \longrightarrow \mathcal{O}_{\mathbf{C}}^{\times}, \quad f \mapsto \exp(f) = \sum_{n=0}^{\infty} \frac{f^n}{n!}$$

where $\mathcal{O}_{\mathbf{C}}$ is the sheaf of holomorphic functions on $\mathbf{C}$ (with additive group structure) and $\mathcal{O}_{\mathbf{C}}^{\times}$ is the sheaf of invertible (i.e. nowhere vanishing) holomorphic functions (with multiplicative group structure). The kernel of the exponential map consists of the locally constant functions with values $2\pi\mathbf{i}k$ for $k \in \mathbf{Z}$ and the sequence

$$0 \longrightarrow \underline{\mathbf{Z}} \xrightarrow{2\pi\mathbf{i}} \mathcal{O}_{\mathbf{C}} \xrightarrow{\exp} \mathcal{O}_{\mathbf{C}}^{\times} \longrightarrow 0$$

is exact. For every open subset $U \subseteq \mathbf{C}$ the induced sequence on sections is left-exact, i.e. the sequence

$$0 \longrightarrow \Gamma(U,\underline{\mathbf{Z}}) \xrightarrow{2\pi\mathbf{i}} \Gamma(U,\mathcal{O}_{\mathbf{C}}) \xrightarrow{\exp} \Gamma(U,\mathcal{O}_{\mathbf{C}}^{\times}) \tag{1}$$

is exact. But there exist open subsets U of $\mathcal{C}$ such that the map $\Gamma(U,\mathcal{O}_{\mathbf{C}}) \xrightarrow{\exp} \Gamma(U,\mathcal{O}_{\mathbf{C}}^{\times})$ is not surjective, i.e. the sequence is not exact anymore if one added a zero to the right. For instance, one can take $U = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$ [reference to be added].

Question 1.1. Can the exact sequence (1) be extended to the right such that the resulting sequence is exact and finishes eventually with zero?

Later in the course we will prove the following answer to the question.

Proposition 1.2. *For every open subset U of $\mathbf{C}$ there exists an abelian group $\mathrm{H}^1(U,\underline{\mathbf{Z}})$ and a group homomorphism $\partial\colon\Gamma(U,\mathcal{O}_{\mathbf{C}}^{\times}) \to \mathrm{H}^1(U,\underline{\mathbf{Z}})$ such that the sequence*

$$0 \longrightarrow \Gamma(U,\underline{\mathbf{Z}}) \xrightarrow{2\pi\mathbf{i}} \Gamma(U,\mathcal{O}_{\mathbf{C}}) \xrightarrow{\exp} \Gamma(U,\mathcal{O}_{\mathbf{C}}^{\times}) \xrightarrow{\partial} \mathrm{H}^1(U,\underline{\mathbf{Z}}) \longrightarrow 0$$

is exact.

The group $\mathrm{H}^1(U,\underline{\mathbf{Z}})$ is called the first **cohomology group** of the sheaf $\underline{\mathbf{Z}}$ on U. Of course, one could just have defined $\mathrm{H}^1(U,\underline{\mathbf{Z}})$ to be the cokernel of the map $\Gamma(U,\mathcal{O}_{\mathbf{C}}) \xrightarrow{\exp} \Gamma(U,\mathcal{O}_{\mathbf{C}}^{\times})$, but this would be cheating. What we want is a general formalism which is functorial in both varying the sheaf and also varying the space. We will prove the following result during the remainder of the course. We will see that $\mathrm{H}^1(U,\underline{\mathbf{Z}})$ only depends on the space U and the sheaf $\underline{\mathbf{Z}}$, but not on $\mathcal{O}_C$ or $\mathcal{O}_C^{\times}$.

Theorem 1.3 (Existence and properties of sheaf cohomology)**.** *Let R be a ring. For every sheaf of R-modules F on a topological space X and for every natural number $n \geq 0$ there exists an R-module $\mathrm{H}^n(X,F)$ such that the following holds:*

1. (Functoriality for the sheaf) *The assignment of objects $\mathrm{Sh}(X,R) \ni F \mapsto \mathrm{H}^n(X,F) \in \mathrm{Mod}(R)$ can be extended to a functor $\mathrm{H}^n(X,-)\colon \mathrm{Sh}(X,R) \to \mathrm{Mod}(R)$.*
2. *For $n = 0$, the functor $\mathrm{H}^0(X,-)$ is canonically isomorphic to the global sections functor $\Gamma(X,-)$.*

(3) (Long exact cohomology sequence) *For every short exact sequence* $0 \to F \to G \to H \to 0$ *of sheaves of* R*-modules on* X *there exist a canonical homomorphisms* $\partial\colon \mathrm{H}^n(X,H) \to \mathrm{H}^{n+1}(X,F)$ *such the the induced sequence*

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X,F) \arrow[r] & \Gamma(X,G) \arrow[r] & \Gamma(X,H) \arrow[dll] \\ & \mathrm{H}^1(X,F) \arrow[r] & \mathrm{H}^1(X,G) \arrow[r] & \mathrm{H}^1(X,H) \arrow[dll] \\ & \mathrm{H}^2(X,F) \arrow[r] & \mathrm{H}^2(X,G) \arrow[r] & \ldots\ldots \end{tikzcd}$$

of R*-modules is exact.*[1]

(4) *For every commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F' \arrow[r] & G' \arrow[r] & H' \arrow[r] & 0 \end{tikzcd}$$

of sheaves of R*-modules on* X *with exact rows we get a commuative diagram*

$$\begin{tikzcd} \mathrm{H}^n(X,F) \arrow[r] \arrow[d] & \mathrm{H}^n(X,G) \arrow[r] \arrow[d] & \mathrm{H}^n(X,H) \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^{n+1}(X,F) \arrow[d] \\ \mathrm{H}^n(X,F') \arrow[r] & \mathrm{H}^n(X,G') \arrow[r] & \mathrm{H}^n(X,H') \arrow[r, "\partial"] & \mathrm{H}^{n+1}(X,F'). \end{tikzcd}$$

(5) (Functoriality for the space) *For every continuous map* $f\colon X \to Y$ *into another topological space* Y *and every sheaf* G *on* Y *there are functorial morphisms*

$$f^*\colon \mathrm{H}^n(Y,G) \longrightarrow \mathrm{H}^n(X,f^{-1}G).$$

(6) (Homotopy invariance) *If* F *is a locally constant sheaf, then the map*

$$p^*\colon \mathrm{H}^n(X,F) \longrightarrow \mathrm{H}^n(X\times[0,1],p^{-1}F)$$

which is induced by the projection $p\colon X\times[0,1] \to X$ *is an isomorphism.*

(7) (Mayer-Vietoris sequence) *If* X *is covered by two open subsets* U *and* V*, then there exist a canonical homomorphism* $\delta\colon \mathrm{H}^n(U\cap V,F|_{U\cap V}) \to \mathrm{H}^{n+1}(X,F)$ *such that the induced sequence*

$$\mathrm{H}^n(X,F) \xrightarrow{\Delta} \mathrm{H}^n(U,F|_U)\oplus\mathrm{H}^n(V,F|_V) \xrightarrow{\pm} \mathrm{H}^n(U\cap V,F|_{U\cap V}) \xrightarrow{\delta} \mathrm{H}^{n+1}(X,F) \xrightarrow{\Delta} \ldots$$

is exact.[2]

Proof. The part (1)–(4) is Theorem 9.3. (5) is Proposition 9.13. (7) is Proposition 9.12. □

[1] The conditions (2) and (3) say that the family $(\mathrm{H}^n(X,-))_{n\geq 0}$ is a δ-functor, cf. Grothendieck's Tohoku paper [Gro57].

[2] Here the map Δ is the map which is in each component the map from (5) and "$\pm$" is the map from (5) in the first component minus the map from (5) in the second component.

2. Abelian categories

The category of abelian sheaves $\mathrm{Sh}(X,\mathbf{Z})$ on a topological space X behaves somewhat similarly to the category Ab of abelian groups, e.g. there are kernels and cokernels and a morphism is a monomorphism (resp. epimorphism) if and only if its kernel (resp. cokernel) is zero. Such phenomena can be described withing any abelian category.

Definition 2.1. (i) A category $\mathcal{A}$ is said to be **additive** iff for every pair of obejects A and B the set $\mathrm{Hom}_{\mathcal{A}}(A,B)$ is an abelian group and such that
   (a) the composition maps are bilinear,
   (b) there exists a final object 0, and
   (c) there exists finite coproducts (called 'direct sums').

(ii) A functor $F\colon\mathcal{A}\to\mathcal{B}$ between additive categories is called **additive** iff for all pair of objects A and B in $\mathcal{A}$ the map

$$F\colon \mathrm{Hom}_{\mathcal{A}}(A,B)\to \mathrm{Hom}_{\mathcal{B}}(F(A),F(B))$$

is a group homomorphism.

(iii) Let $f\colon A\to B$ be a morphism in an additive category $\mathcal{A}$. A morphism $k\colon\ker(f)\to X$ is called **kernel** of f iff $f\circ k=0$ and for every morphism $g\colon X\to A$ such that $f\circ g=0$ there exists a unique morphism $\tilde{g}\colon X\to\ker(f)$ such that $g=k\circ\tilde{g}$.

$$\begin{tikzcd}
& X \arrow[dl, dashed, "\exists!\tilde{g}"'] \arrow[d, "g"] \arrow[dr, "=0"] & \\
\ker(f) \arrow[r, "k"'] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

If it exists, a kernel is unique up to unique isomorphism. Similarly one defines the **cokernel**.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "=0"'] & B \arrow[r] \arrow[d, "g"] & \operatorname{coker}(f) \arrow[dl, dashed, "\exists!\tilde{g}"] \\
& Z &
\end{tikzcd}$$

Then one can define the **image** to be $\mathrm{im}(f):=\ker(B\to\operatorname{coker}(f))$ and the **coimage** $\mathrm{coim}(f):=\operatorname{coker}(\ker(f)\to A)$..

(iv) An additive category is called **abelian** iff every morphism $f\in\mathrm{Hom}_{\mathcal{A}}(A,B)$ has a kernel and a cokernel such that the induced morphism

$$\tilde{f}\colon\mathrm{coim}(f)\to\mathrm{im}(f)$$

is an isomorphism.

Examples 2.2. (i) The category of sets is not additive.

(ii) The categories of rings, ringed spaces, and locally ringed spaces are additive, but not abelian (as kernels of ring homomorphisms are not rings).

(iii) The categories of abelian groups, of R-modules for a ring R, of sheaves of R-modules on a topological space X, and of $\mathcal{O}_X$-modules for a ringed space $(X,\mathcal{O}_X)$ are abelian categories.

Example 2.3. In the category of R-modules for a ring R the kernel of a morphism $f\colon A\to B$ was used to be defined as the subset $\ker(f)=\{a\in A\mid f(a)=0\}$. Then the inclusion map $\ker(f)\hookrightarrow A$ satifies the universal property of the kernel as statet in Definition 2.1 (iii). The similar statements are true for the cokernel, the image, and the coimage $A/\ker(f)$. Convince yourself that this is true!

Caveat 2.4. In an arbitrary abelian category, one can do similar things as in any category of modules over a ring. But a priori one cannot work with *elements* since the obejcts of an abelian category do not need to have an underlying set. So one has to argue instead by using the universal properties (which you should learn). Eventually, one can overcome this issue by using the *Freyd-Mitchell embedding theorem*: every abelian category is equivalent to a full subcategory a category of modules over a suitable ring, cf. [Stacks, Tag 05PL].

Remark 2.5. Abelian categories are the natural framework of homological algebra. For instance, one can talk about exact sequences and the Snake Lemma and the Five Lemma are true in any abelian category.

Theorem 2.6 (Snake Lemma)**.** *Let*

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' & \end{tikzcd}$$

be a commutative diagram in $\mathcal{A}$ with exact rows. Then there exists a morphism $\partial\colon \ker(h) \to \operatorname{coker}(f)$ such that the sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$$

is exact.[3]

Proof. If one can work with elements, e.g. if $\mathcal{A} = \mathrm{Mod}(R)$ for a ring R, then this is a good exercise in diagram chasing, but be aware that the map is well-defined, see this video. For an arbitrary abelian category the proof is more tedious and will be omitted. □

3. Injective objects

In this section let $\mathcal{A}$ be an abelian category.

Reminder 3.1. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between two categories is **left-exact** iff it sends finite limits in $\mathcal{C}$ to limits in $\mathcal{D}$.

Lemma 3.2. *An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ is left-exact if and only if for every exact sequence $0 \to A \to B \to C$ in $\mathcal{A}$ the induced sequence*

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is exact in $\mathcal{B}$.

Proof. If F is left-exact, then it commutes with kernels, hence

$$F(A) = F(\ker(B \to C)) = \ker(F(A) \to F(B))$$

so that the desired exactness follows. Vice versa, if F sends short exact sequences to left-exact sequences, then the argument shows that F preserves kernels. Since additive functors also preserve finite products, it preserves arbitrary finite limits (since every finite limit is a kernel of a map between finite products). □

Remark 3.3. The category $\mathcal{A}^{\mathrm{op}}$ is an abelian category as well. A morphism in $\mathcal{A}^{\mathrm{op}}$ is a kernel, cokernel, monomorphism, or epimorphism if and only if the corresponding morphism in $\mathcal{A}$ is a cokernel, kernel, epimorphism, resp. monomorphism, i.e. everything switches. It is a good yoga with the definitions to convince oneself that this is true.

[3]This is called the *Snake Lemma* since one can draw the resulting exact sequence into the original diagram such that the map ∂ looks like a snake.

Lemma 3.4. *For every object X of $\mathcal{A}$ the functor*

$$\mathrm{Hom}_{\mathcal{A}}(-,X)\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathrm{Ab}, \quad A \mapsto \mathrm{Hom}_{\mathcal{A}}(A,X)$$

is left-exact.

Proof. We use Lemma 3.2 and Remark 3.3, so let $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence in $\mathcal{A}$. We want to show that the induced sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(C,X) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{A}}(B,X) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(A,X)$$

is exact in Ab. Let $h \in \mathrm{Hom}_{\mathcal{A}}(C,X)$ such that $g^*(h) = h \circ g = 0$. Then $h \circ g = 0 \circ g$ and, as g is an epimorphism, $h = 0$. Since $g \circ f = 0$, we infer $f^* \circ g^* = (g \circ f)^* = 0$ and hence $\mathrm{im}(g^*) \subset \ker(f^*)$. For the other inclusion let $h \in \mathrm{Hom}_{\mathcal{A}}(B,X)$ such that $f^*(h) = h \circ f = 0$. Since the morphism $g\colon B \to C$ is a cokernel, there exists a morphism $\bar{h}\colon C \to X$ such that $h = \bar{h} \circ g = g^*(\bar{h})$, i.e. $\ker(f^*) \subset \mathrm{im}(g^*)$. □

Alternative proof (sketch). Assume that there exists an internal hom-functor $\underline{\mathrm{Hom}}_{\mathcal{A}}(-,X)$ which is right-adjoint to an internal tensor functor $(-) \otimes X$. Then $\underline{\mathrm{Hom}}_{\mathcal{A}}(-,X)$ commutes with limits and one can infer that also $\mathrm{Hom}_{\mathcal{A}}(-,X)$ commutes with limits, hence with kernels. □

Definition 3.5. An object I in $\mathcal{A}$ is called **injective** iff the functor $\mathrm{Hom}_{\mathcal{A}}(-,I)$ is exact.

Lemma 3.6. *An object I in $\mathcal{A}$ is injective if and only if for every monomorphism $f\colon X \to Y$ in $\mathcal{A}$ the induced map $f^* := \mathrm{Hom}_{\mathcal{A}}(f,I)\colon \mathrm{Hom}_{\mathcal{A}}(Y,I) \to \mathrm{Hom}_{\mathcal{A}}(X,I)$ is surjective.*

Proof. This is Exercise 4 on Exercise Sheet 5. □

Lemma 3.7. *If I is an injective object of $\mathcal{A}$, then every short exact sequence of the form $0 \to I \xrightarrow{f} X \xrightarrow{g} Y \to 0$ splits (cf. Exercise 5 on Exercise Sheet 5).*

Proof. We want to show that f has a section, i.e. that there exists a morphism $s\colon X \to I$ such that $s \circ f = \mathrm{id}_I$. Since f is a monomorphism, we can choose any preimage of id_I under the morphism surjective morphism $\mathrm{Hom}_{\mathcal{A}}(X,I) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{A}}(I,I)$ (due to Lemma 3.6). □

Lemma 3.8. *For two objects I and J of $\mathcal{A}$, the coproduct $I \oplus J$ is injective if and only if both I and J are injective.*

Proof. In an abelian category, finite coproducts agree with finite products, hence there is an isomorhism of functors $\mathrm{Hom}_{\mathcal{A}}(-,I \oplus J) \cong \mathrm{Hom}_{\mathcal{A}}(-,I) \oplus \mathrm{Hom}_{\mathcal{A}}(-,J)$ and the claim follows. □

Lemma 3.9. *Let $0 \to I' \to I \to I'' \to 0$ be an exact sequence in $\mathcal{A}$ such that I' and I both are injective. Then also I'' is injective.*

Proof. By Lemma 3.7 we have $I \cong I' \oplus I''$ since I' is injective. By Lemma 3.8 get the claim since I is injective. □

4. Right-derived functors

In this section let $\mathcal{A}$ be an abelian category.

Definition 4.1. We say that $\mathcal{A}$ **has enough injective objects** iff for every object A there exists an injective object I and a monomorphism $A \to I$.

Proposition 4.2. *Let R be a ring (commutative, with unit, as always). Then the category $\mathrm{Mod}(R)$ of R-modules has enough injective objects.*

Proof. Outsourced, see [Stacks, Tag 01D8]. □

Theorem 4.3. *Let $(X,\mathcal{O}_X)$ be a ringed space. Then the category* $\mathrm{Mod}(\mathcal{O}_X)$ *of $\mathcal{O}_X$-modules has enough injective objects.*

Proof. Let F be an $\mathcal{O}_X$-module. For every $x \in X$ the stalk F_x is an $\mathcal{O}_{X,x}$-module, hence there exist monomorphisms $F_x \to I_x$ for suitable injective $\mathcal{O}_{X,x}$-modules I_x.

The module I_x is a sheaf on the singleton $\{x\}$ and yields via the inclusion $j_x : \{x\} \to X$ an $\mathcal{O}_X$-module $j_{x,*}I_x$. Consider the product $I := \prod_{x\in X} j_{x,*}I_x$. For every $\mathcal{O}_X$-module G we have by the universal property of the product and by the adjunction $(j_x^*, j_{x,*})$ that

$$\mathrm{Hom}_{\mathcal{O}_X}(G,I) = \prod_{x\in X} \mathrm{Hom}_{\mathcal{O}_X}(G, j_{x,*}I_x) = \prod_{x\in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(G_x, I_x).$$

For $G = F$ we get an injective morphismus $F \to I$ induced by the injective morphisms $F_x \to I_x$. Since the I_x are injective, the functors $\mathrm{Hom}_{\mathcal{O}_{X,x}}(-,I_x)$ are exact. By the identifications above, also the functor $\mathrm{Hom}_{\mathcal{O}_X}(-,I)$ is exact, hence I is injective. Furthermore, the morphism $F \to I$ is a monomorphism since it is a monomorphism on every stalk. □

Corollary 4.4. *For every topological space X and every ring R the category* $\mathrm{Sh}(X,R)$ *of sheaves of R-modules on X has enough injective objects.*

Proof. The constant sheaf $\underline{R}$ yields a ringed space $(X,\underline{R})$ and $\mathrm{Sh}(X,R) = \mathrm{Mod}(\underline{R})$. □

Remark 4.5. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact additive functor into an abelian category $\mathcal{B}$. Then for every short exact sequence $0 \to I \to B \to C \to 0$ in $\mathcal{A}$ such that I is injective, the induced sequence

$$0 \longrightarrow F(I) \longrightarrow F(B) \longrightarrow F(C) \to 0$$

is exact by Lemma 3.7. One can say that injective objects behave better with respect to left-exact additive functors than arbitrary objects . But if $\mathcal{A}$ has enough injectives, then we can "resolve" any object in $\mathcal{A}$ by injective objects to be better off.

Definition 4.6. Let A be an object of $\mathcal{A}$. An **injective resolution** of A in $\mathcal{A}$ is an exact sequence $I^0 \to I^1 \to I^2 \to \dots$ in $\mathcal{A}$ where all obejcts I^n are injective in $\mathcal{A}$ together with a morphism $A \to I^0$ which is a kernel of the morphism $I^0 \to I^1$. Hence we have an exact sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

Proposition 4.7. *Assume that $\mathcal{A}$ has enough injective objects. Then every object in $\mathcal{A}$ admits an injective resolution.*

Proof. This is Exercise 1 on Exercise Sheet 5. □

Construction 4.8. Assume that $\mathcal{A}$ has enough injectives objects and let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact additive functor into an abelian category $\mathcal{B}$. For an object A in $\mathcal{A}$ we choose an injective resolution $I^0 \xrightarrow{d_I^0} I^1 \xrightarrow{d_I^1} I^2 \xrightarrow{d_I^2} \dots$. Forgetting A and applying the functor F the induced sequence

$$0 \xrightarrow{F(d_I^{-1})} F(I^0) \xrightarrow{F(d_I^0)} F(I^1) \xrightarrow{F(d_I^1)} F(I^2) \xrightarrow{F(d_I^2)} \dots$$

may not be exact anymore, but it is still a complex, i.e. $F(d_I^{n+1}) \circ F(d_I^n) = 0$ for every $n \geq 0$ (cf. Definition 4.12 below). Thus we can define for every $n \in \mathbf{N}$ the cohomology group

$$\mathrm{R}^n_{I^\bullet} F(A) := \mathrm{H}^n(F(I^\bullet)) = \frac{\ker(F(d_I^n))}{\mathrm{im}(F(d_I^{n-1}))}.$$

Theorem 4.9 (Existence and properties of right-derived functors)**.** *Assume that $\mathcal{A}$ has enough injective objects and let $F\colon \mathcal{A} \to \mathcal{B}$ a left-exact additive functor into another abelian category $\mathcal{B}$.*

(1) (Uniqueness) *The definition of* $\mathrm{R}^n_{I^\bullet} F(A)$ *is independent of the choice of the injective resolution so that we can just write* $\mathrm{R}^n F(A)$.

(2) (Functoriality) *For every* $n \in \mathbf{N}$ *we get an additive functor*

$$\mathrm{R}^n F \colon \mathcal{A} \to \mathcal{B}, \quad A \mapsto \mathrm{R}^n F(A).$$

(3) $F \cong \mathrm{R}^0 F$.

(4) (Long exact sequence) *For every short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

and for every $n \in \mathbf{N}$ *there exists a canonical homomorphism*

$$\partial^n \colon \mathrm{R}^n F(C) \to \mathrm{R}^{n+1} F(A)$$

and a long exact sequence

$$\ldots \longrightarrow \mathrm{R}^n F(A) \xrightarrow{\mathrm{R}^n F(f)} \mathrm{R}^n F(B) \xrightarrow{\mathrm{R}^n F(g)} \mathrm{R}^n F(C) \xrightarrow{\partial^n} \mathrm{R}^{n+1} F(A) \longrightarrow \ldots$$

(5) (Functoriality for LES) *For every commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

in $\mathcal{A}$ *with exact rows and every* $n \in \mathbf{N}$ *we get a commuative diagram*

$$\begin{tikzcd}
\mathrm{R}^n F(A) \arrow[r] \arrow[d] & \mathrm{R}^n F(B) \arrow[r] \arrow[d] & \mathrm{R}^n F(C) \arrow[r, "\partial^n"] \arrow[d] & \mathrm{R}^{n+1} F(A) \arrow[d] \\
\mathrm{R}^n F(A') \arrow[r] & \mathrm{R}^n F(B') \arrow[r] & \mathrm{R}^n F(C') \arrow[r, "\partial^n"] & \mathrm{R}^{n+1} F(A').
\end{tikzcd}$$

Proof. The first part begins on page 11 and the second part begins on page 12. □

Definition 4.10. If $\mathcal{A}$ has enough injective objects, then we call for a left-exact additive functor $F \colon \mathcal{A} \to \mathcal{B}$ into an abelian category $\mathcal{B}$ and for $n \in \mathbf{N}$ the induced functor $\mathrm{R}^n F \colon \mathcal{A} \to \mathcal{B}$ the **n-th right-derived functor** of F.

Remark 4.11. In a dual manner, for a *right*-exact functor G one can define its *left*-derived functors $\mathrm{L}^n G$; instead of injective objects and injective resolutions one works with *projective* objects and *projective resolutions*. See Exercise 3 on Exercise Sheet 5. This will also be subject to another exercise.

Definition 4.12. (i) A **(cochain) complex** in $\mathcal{A}$ is a sequence of morphisms

$$\ldots \longrightarrow A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \xrightarrow{d_A^{n+1}} \ldots$$

such that $d_A^n \circ d_A^{n-1} = 0$ for every $n \in \mathbf{Z}$. We write $(A^\bullet, d_A)$ for such a complex; sometimes the morphism d_A will be omitted in the notation.

(ii) A **morphism of (cochain) complexes** $f^\bullet \colon (A^\bullet, d_A) \to (B^\bullet, d_B)$ in $\mathcal{A}$ is a family of morphisms $(f^n \colon A^n \to B^n)_{n \in \mathbf{Z}}$ such that $f^{n+1} \circ d_A^n = d_B^n \circ f^n$ for all $n \in \mathbf{Z}$. This means that the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"] & \ldots \\
\ldots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"'] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r, "d_B^{n+1}"'] & \ldots
\end{tikzcd}$$

is commutative.

(iii) We denote by $\mathrm{Ch}^\bullet(\mathcal{A})$ the **category of (cochain) complexes** in $\mathcal{A}$.[4]

(iv) For a complex $(A^\bullet, d_A)$ in $\mathcal{A}$ we define for every $n \in \mathbf{Z}$ the **n-th cohomology group**

$$\mathrm{H}^n(A^\bullet, d_A) := \frac{\ker(d_A^n)}{\mathrm{im}(d_A^{n-1})}.$$

(v) Let $f^\bullet\colon (A^\bullet, d_A) \to (B^\bullet, d_B)$ be a morphism of complexes in $\mathcal{A}$. We say that f is **nullhomotopic** (and write $f \simeq 0$) iff for every $n \in \mathbf{Z}$ there exists a morphism $s^n\colon A^n \to B^{n-1}$ such that $d_B^{n-1} \circ s^n + s^{n+1} \circ d_A^n = f^n$. Here is the situation in a diagram:[5]

$$\begin{tikzcd} A^{n-1} \arrow[r, "d_A^{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[dl, "s^n"'] \arrow[d, "f^n"] & A^{n+1} \arrow[dl, "s^{n+1}"'] \\ B^{n-1} \arrow[r, "d_B^{n-1}"'] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \end{tikzcd}$$

The family $(s_n\colon A^n \to B^{n-1})_{n\in\mathbf{Z}}$ is called a **homotopy**.[6]

(vi) Let $f, g\colon (A^\bullet, d_A) \to (B^\bullet, d_B)$ be two morphisms of complexes. We say that f is **homotopic to** g (and write $f \simeq g$) iff the morphism $f - g$ is nullhomotopic.

Example 4.13. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a split-exact sequence in $\mathcal{A}$, i.e. there exist morphisms $r\colon B \to A$ and $s\colon C \to B$ such that $r \circ f = \mathrm{id}_A$ and $g \circ s = \mathrm{id}_C$. Considered as a complex, the identity on it is nullhomotopic:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "g"] \arrow[d, "\mathrm{id}_B"] \arrow[dl, "r"'] & C \arrow[r] \arrow[d, "\mathrm{id}_C"] \arrow[dl, "s"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

This follows from Exercise 5 on Exercise Sheet 5.

Lemma 4.14. *(i) A morphism of complexes $f\colon (A^\bullet, d_A) \to (B^\bullet, d_B)$ induces for every $n \in \mathbf{Z}$ a morphism on the n-th cohomology groups*

$$\mathrm{H}^n(f)\colon \mathrm{H}^n(A^\bullet, d_A) \to \mathrm{H}^n(B^\bullet, d_B)$$

which yields an additive functor $\mathrm{H}^n(-)\colon \mathrm{Ch}^\bullet(\mathcal{A}) \to \mathcal{A}$.

(ii) Let $f, g\colon (A^\bullet, d_A) \to (B^\bullet, d_B)$ be two morphisms of complexes which are homotopic. Then they induce the same morphism on cohomology groups, i.e. for every $n \in \mathbf{Z}$ the induced morphisms $\mathrm{H}^n(f)\colon \mathrm{H}^n(A) \to \mathrm{H}^n(B)$ and $\mathrm{H}^n(g)\colon \mathrm{H}^n(A) \to \mathrm{H}^n(B)$ are equal.

Proof. (i) is a straightforward exercise. For (ii) is suffices to show that $\mathrm{H}^n(f) = 0$ if $f \simeq 0$ (by additivity) so that we may assume that $g = 0$. Let $(s_n\colon A^n \to B^{n-1})_{n\in\mathbf{Z}}$ be a homotopy such that $d_B^{n-1} \circ s^n + s^{n+1} \circ d_A^n = f$ for every $n \in \mathbf{Z}$. Restricting to $\ker(d_A^n)$ we get that

$$f|_{\ker(d_A^n)} = (d_B^{n-1} \circ s^n + s^{n+1} \circ d_A^n)|_{\ker(d_A^n)} = (d_B^{n-1} \circ s_n)|_{\ker(d_A^n)}$$

so that $\mathrm{im}(f|_{\ker(d_A^n)}) \subset \mathrm{im}(d_B^{n-1})$ which implies that $\mathrm{H}^n(f)$ is the zero morphism. □

[4] The category $\mathrm{Ch}^\bullet(\mathcal{A})$ can also be desribed as a functor category, see Exercise 6 on Exercise Sheet 5.

[5] The condition $d_B^{n-1} \circ s^n + s^{n+1} \circ d_A^n = f^n$ displays $f^\bullet$ in terms of the images of d_A and d_B which yields the resulting morphism on cohomology to be zero, see Lemma 4.14.

[6] Yes, there is a relation with the notion of a homotopy between maps of topological spaces.

Proposition 4.15. *Let $f : A \to B$ be a morphism in $\mathcal{A}$, $A \hookrightarrow I^\bullet$ any resolution*[7], *$B \to J^\bullet$ a complex with such that J^n is an injective object for every $n \in \mathbf{N}$.*

(i) For every $n \in \mathbf{N}$ there exists a morphism $f^n : I^n \to J^n$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & I^0 \arrow[r] \arrow[d, "f^0"'] & I^1 \arrow[r] \arrow[d, "f^1"'] & \ldots \\ B \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \ldots \end{tikzcd}$$

commutes. In particular, we get a morphism of complexes $f^\bullet : I^\bullet \to J^\bullet$ (by forgetting the morphism $f : A \to B$).

(ii) The morphism of complexes $f^\bullet$ in (i) is unique up to homotopy.

Proof. (i) Since J^0 is injective and $A \to I^0$ is a monomorphism, the composition $A \xrightarrow{f} B \to J^0$ extends to a morphism $f^0 : I^0 \to J^0$ such that the resulting square commutes. Now we proceed by induction and assume that the morphism f^{n-1} and f^n are already constructed (setting $f^{-1} = f$). Then the existence of a compatible morphism f^{n+1} follows since J^{n+1} is injective as indicated in the diagram

$$\begin{tikzcd} I^{n-1} \arrow[r, "d_I^{n-1}"] \arrow[d, "f^{n-1}"'] & I^n \arrow[r] \arrow[d, "f^n"'] & \operatorname{coker}(d_I^{n-1}) \arrow[r, hook] \arrow[d, "\bar{f}^n"'] & I^{n+1} \arrow[d, dotted, "\exists f^n"] \\ J^{n-1} \arrow[r, "d_J^{n-1}"] & J^n \arrow[r] & \operatorname{coker}(d_J^{n-1}) \arrow[r] & J^{n+1}. \end{tikzcd}$$

(ii) We have to show that $f^\bullet \simeq g^\bullet$ for any other morphism of complexes $g^\bullet : I^\bullet \to J^\bullet$. By additivity it suffices to show that $f^\bullet \simeq 0$ if $f = 0$. So we have to construct a homotopy $(s^n : I^n \to J^{n-1})_{n \in \mathbf{Z}}$ where $I^n = 0 = J^n$ for $n < 0$. We set $s^0 = 0$ and proceed by induction. So we assume that we already have constructed $(s^i)_{0 \le i \le n}$ and want shall construct s^{n+1} as indicated in the diagram:

$$\begin{tikzcd} I^{n-1} \arrow[r, "d_I^{n-1}"] & I^n \arrow[r, "d_I^n"] \arrow[dl, "s^n"] \arrow[d, "f^n"] & I^{n+1} \arrow[dl, dotted, "s^{n+1}"] \\ J^{n-1} \arrow[r, "d_J^{n-1}"'] & J^n \arrow[r, "d_J^n"'] & J^{n+1} \end{tikzcd}$$

By induction hypothesis we have that $d_J^n \circ s^{n-1} + s^n \circ d_I^{n-1} = f^{n-1}$ so that we have

$$(f^n - d_J^{n-1} \circ s^n) \circ d_I^{n-1} = f^n \circ d_I^{n-1} - d_J^{n-1} \circ s^n \circ d_I^{n-1} = f^n \circ d_I^{n-1} - d_J^{n-1} \circ f^{n-1} = 0$$

where the last equality holds since $f^\bullet$ is a moprhism of complexes. Hence the morphism $f^n - d_J^{n-1} \circ s^n$ factors over $\operatorname{coker}(d_I^{n-1})$ so that we get a morphism $t : \operatorname{coker}(d_I^{n-1}) \to J^n$. Since the complex $I^\bullet$ is exact, the morphism $d_I^n : I^n \to I^{n+1}$ factors as

$$I^n \twoheadrightarrow \operatorname{coker}(d_I^{n-1}) = I^n / \operatorname{im}(d_I^{n-1}) \cong I^n / \ker(d_I^n) = \operatorname{im}(d_I^n) \hookrightarrow I^{n+1}.$$

Using that J^n is injective we see that the morphism t extends to the desired morphism s^{n+1} as indicated in the diagram:

$$\begin{tikzcd} I^n \arrow[r, two heads] \arrow[rr, bend left, "d_I^n"] \arrow[d, "f^n - d_J^{n-1} \circ s^n"'] & \operatorname{coker}(d_I^{n-1}) \arrow[r, hook] \arrow[dl, "t"'] & I^{n+1} \arrow[dll, dotted, "s^{n+1}"] \\ J^n & & \end{tikzcd}$$

[7]That is an exact sequence $0 \to A \to I^0 \to I^1 \to \ldots$.

By construction, we have $s^{n+1} \circ d_I^n = f^n - d_J^{n-1} \circ s^n$, hence the homotopy condition holds true. By induction, $f^\bullet \simeq 0$ which was to show. □

Proof of Theorem 4.9 (1)-(3). (1) *(Uniqueness)* Let $A \hookrightarrow I^\bullet$ and $A \hookrightarrow J^\bullet$ be two injective resolutions. By Proposition 4.15 (i) applied twice to id_A there exist morphisms of complexes $f: I^\bullet \to J^\bullet$ and $g: J^\bullet \to I^\bullet$. By Proposition 4.15 (ii) the composition $g^\bullet \circ f^\bullet$ is homotopic to $\mathrm{id}_{I^\bullet}$ since they both extend id_A along $I^\bullet$; analogously, $f^\bullet \circ g^\bullet \simeq \mathrm{id}_{J^\bullet}$. By funtoriality, we have homotopies $F(\mathrm{id}_{I^\bullet}) \simeq F(g^\bullet) \circ F(f^\bullet)$ and $F(\mathrm{id}_{J^\bullet}) \simeq F(f^\bullet) \circ F(g^\bullet)$. By Lemma 4.14 we get that $\mathrm{H}^n(F(f^\bullet))$ and $\mathrm{H}^n(F(g^\bullet))$ are inverse isomorphisms.

(2) *(Functoriality)* Le $f: A \to B$ be morphism in $\mathcal{A}$. Choose injective resolutions $A \hookrightarrow I^\bullet$ and $B \hookrightarrow J^\bullet$. By Proposition 4.15 there exists a morphism of complexes $f^\bullet: I^\bullet \to J^\bullet$ which is unique up to homotopy. By Lemma 4.14 this induces a unique morphism on cohomology groups. Convince yourself that this is compatible with compositions.

(3) $(F \cong \mathrm{R}^0 F)$ Let $A \xrightarrow{\varphi} I^0 \xrightarrow{d_I^0} I^1 \xrightarrow{d_I^1} \ldots$ be an injective resolution. Then the incuced morphism $A \xrightarrow{\tilde{\varphi}} \ker(I^0 \xrightarrow{d_I^0} I^1)$ is an isomorphism. Since F is left exact, we infer that the induced morphism

$$F(A) \xrightarrow{F(\tilde{\varphi})} F\big(\ker(I^0 \xrightarrow{d_I^0} I^1)\big) \cong \ker\big(F(I^0) \xrightarrow{F(d_I^0)} F(I^1)\big) = \mathrm{H}^0(F(I^\bullet))$$

is an isomorphism. This is also compatible with morphisms so that we get an isomorphism of fuctors. □

Definition 4.16. A **short exact sequence of complexes** in $\mathcal{A}$ is a sequence

$$0 \longrightarrow A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \longrightarrow 0$$

of morphism of complexes in $\mathcal{A}$ such that for every $n \in \mathbf{Z}$ the induced sequence

$$0 \longrightarrow A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \longrightarrow 0$$

is exact.[8]

Proposition 4.17. *(i) Let $0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$ be a short exact sequence of complexes in $\mathcal{A}$. Then for every $n \in \mathbf{Z}$ there exists morphism*

$$\partial^n : \mathrm{H}^n(C^\bullet) \longrightarrow \mathrm{H}^{n+1}(A^\bullet)$$

such that the sequence

$$\ldots \to \mathrm{H}^{n-1}(C^\bullet) \xrightarrow{\partial^{n-1}} \mathrm{H}^n(A^\bullet) \xrightarrow{\mathrm{H}^n(f^\bullet)} \mathrm{H}^n(B^\bullet) \xrightarrow{\mathrm{H}^n(g^\bullet)} \mathrm{H}^n(C^\bullet) \xrightarrow{\partial^n} \mathrm{H}^{n+1}(A^\bullet) \to \ldots$$

is exact.

(ii) For every commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tilde{A}^\bullet \arrow[r] & \tilde{B}^\bullet \arrow[r] & \tilde{C}^\bullet \arrow[r] & 0 \end{tikzcd}$$

[8] The category of complexes $\mathrm{Ch}(\mathcal{A})$ is also an abelian category and this notion of a short exact sequence coincides with the notion of a short exact sequence internal to the abelian category $\mathrm{Ch}(\mathcal{A})$.

of complexes in $\mathcal{A}$ with exact rows and every $n \in \mathbf{N}$ we get a commuative diagram

$$\begin{tikzcd}
\mathrm{H}^n(A^\bullet) \arrow[r] \arrow[d] & \mathrm{H}^n(B^\bullet) \arrow[r] \arrow[d] & \mathrm{H}^n(C^\bullet) \arrow[r, "\partial^n"] \arrow[d] & \mathrm{H}^{n+1}(A^\bullet) \arrow[d] \\
\mathrm{H}^n(\tilde{A}\bullet) \arrow[r] & \mathrm{H}^n(\tilde{B}^\bullet) \arrow[r] & \mathrm{H}^n(\tilde{C}^\bullet) \arrow[r, "\partial^n"] & \mathrm{H}^{n+1}(\tilde{A}^\bullet).
\end{tikzcd}$$

Proof. (i) For every $n \in \mathbf{Z}$ we have an exact sequence

$$0 \longrightarrow \mathrm{H}^n(A^\bullet) \overset{\ker(\tilde{d}_A^n)}{\longrightarrow} \operatorname{coker}(d_A^{n-1}) \overset{\tilde{d}_A^n}{\longrightarrow} \ker(d_A^{n+1}) \overset{\operatorname{coker}(\tilde{d}_A^n)}{\longrightarrow} \mathrm{H}^{n+1}(A^\bullet) \longrightarrow 0$$

where $d_A^\bullet$ is the differential on $A^\bullet$ (and the same holds for $B^\bullet$ and $C^\bullet$, of course). Applying the Snake Lemma (Theorem 2.6) to the commutative diagram

$$\begin{tikzcd}
 & \operatorname{coker}(d_A^{n-1}) \arrow[r, "\tilde{f}^{n-1}"] \arrow[d, "\tilde{d}_A^n"] & \operatorname{coker}(d_B^{n-1}) \arrow[r, "\tilde{g}^{n-1}"] \arrow[d, "\tilde{d}_B^n"] & \operatorname{coker}(d_C^{n-1}) \arrow[r] \arrow[d, "\tilde{d}_C^n"] & 0 \\
0 \arrow[r] & \ker(d_A^{n+1}) \arrow[r, "f^{n+1}"] & \ker(d_B^{n+1}) \arrow[r, "g^{n+1}"] & \ker(d_C^{n+1}) &
\end{tikzcd}$$

yields the desired long exact sequence.

(ii) This is straightforward, but tedious to write down. Basically it follows from a functorial version of the Snake Lemma ("functorial" for morphism of exact sequences). □

Lemma 4.18. *Assume that $\mathcal{A}$ has enough injective objects and let $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ be a short exact sequence in $\mathcal{A}$. Then there exist a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^\bullet \arrow[r] & J^\bullet \arrow[r] & K^\bullet \arrow[r] & 0
\end{tikzcd}$$

such that the vertical morphism are injective resolutions and such that the bottom line is a short exact sequence of complexes.

Proof. This remains as an exercise. Hint: choose first injective resolutions $A \hookrightarrow I^\bullet$ and $C \hookrightarrow K^\bullet$; then you can define J^n as $I^n \oplus K^n$, but you have to take care about the morphisms.x □

Proof of Theorem 4.9 (4)-(5). (4) *(Long exact sequence)* Given a short exact sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ in $\mathcal{A}$, there exists a short exact sequence of injective resolutions

$$0 \to I^\bullet \overset{f^\bullet}{\to} J^\bullet \overset{g^\bullet}{\to} K^\bullet \to 0$$

by Lemma 4.18. By Lemma 3.7, for every $n \in \mathbf{N}$ the sequence

$$\begin{tikzcd}
0 \arrow[r] & I^n \arrow[r, shift left, "f^n"] & J^n \arrow[l, shift left, "r^n"] \arrow[r, shift left, "g^n"] & K^n \arrow[l, shift left, "s^n"] \arrow[r] & 0
\end{tikzcd}$$

splits since I^n is injective. Since the functor F is left exact, every induced sequence

$$0 \to F(I^n) \overset{F(f^n)}{\to} F(J^n) \overset{F(g^n)}{\to} F(K^n) \to 0$$

is exact as well.[9] Hence the induced sequene

$$0 \to F(I^\bullet) \overset{F(f^\bullet)}{\to} F(J^\bullet) \overset{F(g^\bullet)}{\to} F(K^\bullet) \to 0$$

[9]It is a general statement that left-exact functors send *split-exact* sequences to short exact sequences; the same holds for right-exact functors.

is a short exact sequence of complexes so that we can apply Proposition 4.17 and get the desired long exact sequence.

(5) *(Functoriality of LES)* This follows from the respective functoriality statements of the statements used in the proof of (4). □

Definition 4.19. For every object X of $\mathcal{A}$, the covariant functor $\mathrm{Hom}_{\mathcal{A}}(X,-)$ is left-exact.[10] For $n \in \mathbf{N}$ call its n-th right-derived functor

$$\mathrm{Ext}^n_{\mathcal{A}}(X,-) := \mathrm{R}^n \mathrm{Hom}_{\mathcal{A}}(X,-) \colon \mathcal{A} \longrightarrow \mathrm{Ab}, \quad A \mapsto \mathrm{Ext}^n_{\mathcal{A}}(X,A)$$

the **n-th Ext-functor**.

Definition 4.20. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact functor into an abelian category $\mathcal{B}$ and assume that $\mathcal{A}$ has enough injective objects. An object J of $\mathcal{A}$ is called ***F*-acyclic** iff for every $n > 0$ on has $\mathrm{R}^n F(J) = 0$.

Example 4.21. By design, injective objetcs are F-acyclic.

Definition 4.22. Let A be an object of $\mathcal{A}$. An ***F*-acyclic resolution** of A is an exact sequence $0 \to A \overset{\varphi}{\to} J^0 \overset{d_J^0}{\to} J^1 \overset{d_J^1}{\to} \dots$ such that J^n is F-acyclic for every $n \in \mathbf{N}$. We shall write $A \overset{\varphi}{\hookrightarrow} J^\bullet$ for such a resolution.

Proposition 4.23. *Let A be an object of $\mathcal{A}$ and let $A \overset{\varphi}{\hookrightarrow} J^\bullet$ be an F-acyclic resolution of A and let $n \in \mathbf{N}$. Then there exists a canonical isomorphism $\mathrm{R}^n F(A) \cong \mathrm{H}^n(F(J^\bullet))$.*

Proof. We construct an injective resolution $A \hookrightarrow I^\bullet$ together with a monomorphism of complexes $f^\bullet\colon J^\bullet \hookrightarrow I^\bullet$ extending the identity on A. Any momomorphism $f^0\colon J^0 \hookrightarrow I^0$ into an injective object I^0 yields $A \overset{\varphi}{\hookrightarrow} J^0 \overset{f^0}{\hookrightarrow} I^0$ so that we obtain a commutative square

$$\begin{tikzcd}
A \arrow[r, hook, "\varphi"] \arrow[d, "\mathrm{id}_A"'] & J^0 \arrow[d, hook, "f^0"] \\
A \arrow[r, hook, "f^0\circ\varphi"'] & I^0
\end{tikzcd}$$

Note that $\mathrm{im}(f^0 \circ \varphi) = f^0(\mathrm{im}(\varphi))$ by design.

Proceeding by induction we assume that we have already constructed compatible monomorphisms $f^k\colon J^k \to I^k$ into injective objects I^k for $0 \le k \le n$ such that $\mathrm{im}(d_I^{n-1}) = f^n(\mathrm{im}(d_J^{n-1}))$.

Consider the pushout square

$$\begin{tikzcd}
J^n \arrow[r, "d_J^n"] \arrow[d, hook, "f^n"] & J^{n+1} \arrow[d, hook, "i"] \\
I^n \arrow[r] & I^n \oplus_{J^n} J^{n+1}.
\end{tikzcd}$$

Since f^n is a monomorphism, also i is a monomorphism (exercise). We can further find a monomorphism $j\colon I^n \oplus_{J^n} J^{n+1} \hookrightarrow I^{n+1}$ into an injective object I^{n+1} so that we obtain a monomorphism $f^{n+1} := j \circ i\colon J^{n+1} \to I^{n+1}$ and a morphism $d_I^n\colon I^n \to I^{n+1}$ as indicated in the

[10] Cf. Exercise 3 on Exercise Sheet 5.

commutative diagram

$$\begin{tikzcd}
J^{n-1} \arrow[r, "d_J^{n-1}"] \arrow[d, hook, "f^{n-1}"'] & J^n \arrow[r, "d_J^n"] \arrow[d, hook, "f^n"'] & J^{n+1} \arrow[d, hook, "i"'] \arrow[dr, hook, dotted, "f^{n+1}"] & \\
I^{n-1} \arrow[r, "d_I^{n-1}"] & I^n \arrow[r, "k"] \arrow[rr, dotted, bend right, "d_I^n"'] & I^n \oplus_{J^n} J^{n+1} \arrow[r, hook, "j"] & I^{n+1}.
\end{tikzcd}$$

We have to show exactness at I^n, i.e. that $\operatorname{im}(d_I^{n-1}) = \ker(d_I^n)$. This follows from the equalities

$$\operatorname{im}(d_I^{n-1}) \overset{(1)}{=} f^n(\operatorname{im}(d_J^{n-1})) \overset{(2)}{=} f^n(\ker(d_J^n)) \overset{(3)}{=} \ker(k) \overset{(4)}{=} \ker(j \circ k) = \ker(d_I^n)$$

where (1) is part of the induction hypothesis, (2) follows from exactness of $J^\bullet$, (3) is a general statement for pushouts, and (4) follows since j is a monomorphism.

Hence we have a monomorphism of complexes $f^\bullet : J^\bullet \hookrightarrow I^\bullet$ into a complex consisting of injective objects. Similarly as in the proof of Proposition 4.15 one can show that the resulting morphism $f^\bullet$ is unique up to homotopy. It remains to show that for every $n \in \mathbf{N}$ the induced morphism

$$\mathrm{H}^n(F(f^\bullet)) : \mathrm{H}^n(F(J^\bullet)) \longrightarrow \mathrm{H}^n(F(I^\bullet)) = \mathrm{R}^n F(A)$$

is an isomorphism.

Together with the cokernel complex $K^\bullet := \operatorname{coker}(f^\bullet : J^\bullet \hookrightarrow I^\bullet)$ we get an exact sequence of complexes $0 \to J^\bullet \xrightarrow{f^\bullet} I^\bullet \xrightarrow{g^\bullet} K^\bullet \to 0$. For every $n \in \mathbf{N}$ we have an exact sequence

$$0 \longrightarrow F(J^n) \overset{F(f^n)}{\longrightarrow} F(I^n) \overset{F(g^n)}{\longrightarrow} F(K^n) \longrightarrow \mathrm{R}^1 F(J^n) = 0$$

so that the induced sequence of complexes

$$0 \longrightarrow F(J^\bullet) \overset{F(f^\bullet)}{\longrightarrow} F(I^\bullet) \overset{F(g^\bullet)}{\longrightarrow} F(K^\bullet) \longrightarrow 0$$

is exact. Thus we get a long exact cohomology sequence

$$\ldots \to \mathrm{H}^{n-1}(F(K^\bullet)) \overset{\partial}{\longrightarrow} \mathrm{H}^n(F(J^\bullet)) \overset{\mathrm{H}^n(F(f^\bullet))}{\longrightarrow} \mathrm{H}^n(F(I^\bullet)) \overset{\mathrm{H}^n(F(g^\bullet))}{\longrightarrow} \mathrm{H}^n(F(K^\bullet)) \to \ldots$$

by Proposition 4.17.

Hence it suffices to show that for every $n \in \mathbf{N}$ the object $\mathrm{H}^n(F(K^\bullet))$ is zero.

Consider the shifted complex $K[1]^\bullet$ with $K[1]^n := K^{n+1}$. The differential d_K yields a morphism of complexes $d_K^\bullet : \mathrm{K}^\bullet \to K[1]^\bullet$ which is in degree n given by the morphism $d_K^n : K^n \to K^{n+1}$.

We set $L^\bullet := \operatorname{coker}(d_K^\bullet)$. Since $K^\bullet$ is exact, we have that

$$L^n = \operatorname{coker}\big(K^n \overset{d_K^n}{\longrightarrow} K^{n+1}\big) \cong K^{n+1}/\operatorname{im}(d_K^n) \cong \ker\big(K^{n+1} \overset{d_K^{n+1}}{\longrightarrow} K^{n+2}\big)$$

Hence $0 \to L^\bullet \to K[1]^\bullet \to L[1]^\bullet \to 0$ is an exact sequence of complexes. Since $L^0 = K^0$ one can check inductively that all the L^n are F-acyclic as well.[11] Thus the sequence

$$0 \longrightarrow F(L^\bullet) \longrightarrow F(K[1]^\bullet) \longrightarrow F(L[1]^\bullet) \longrightarrow 0$$

is an exact sequence of complexes (as $\mathrm{R}^1 F(L^n) = 0$ for every $n \geq 1$). The long exact cohomology sequence then shows that $\mathrm{H}^n(F(K^\bullet)) = 0$ as desired. Is this last step clear to you? □

[11]At this point the argument needs that the sequence $0 \to K^0 \to K^1$ is exact. Hence this is not valid anymore for $J^\bullet$ or $I^\bullet$ which also are F-acyclic. Otherwise, there would not be any higher derived functors at all.

Remark 4.24. The assertion in Proposition 4.23 tells us that we can compute the derived functor $\mathrm{R}^n F$ also by using F-acyclic resolutions instead of only injective resolutions. This can help in practice since we have a greater pool of objects from which we can construct our resolution. Even if $\mathcal{A}$ had not enough injective objects, we could define right-derived functors $\mathrm{R}^n F$ in case there are at least enough F-acyclic objects.

5. Application: injective abelian groups

In this section we want to characterise injective abelian groups. Afterwards we use this characterisation to compute examples of injective resolutions and then of the derived hom-functor.

Theorem 5.1 (Baer's criterion)**.** *Let R be a ring. Then an R-module I is an injective object in* $\mathrm{Mod}(R)$ *if and only if for every ideal $A \subset R$ and every morphism $A \to I$ there exists an extension $R \to I$.*

This means that we can check whether an R-module is injective with inclusions of ideals instead of arbitrary monomorphisms.

Proof sketch. Let $M \hookrightarrow N$ be a monomorphism of R-modules and let $f\colon M \to I$ be any morphism. We want to show that f extends to a morphism $g\colon N \to I$, i.e. $g|_M = f$.

We identify M with its image consider the set

$$\mathrm{Ext}_f(N) := \{(\tilde{N}, \tilde{g}) \mid M \subset \tilde{N} \subset N \;\&\; \tilde{g}\colon \tilde{N} \to I \text{ s.t. } \tilde{g}|_M = f\}.$$

We equip $\mathrm{Ext}_f(N)$ with the partial order given by $(\tilde{N}_1, \tilde{g}_1) \le \tilde{N}_2, \tilde{g}_2)$ iff $\tilde{N}_1 \subset \tilde{N}_2$ and $\tilde{g}_2|_{\tilde{N}_1} = \tilde{g}_1$. Check that the poset $(\mathrm{Ext}_f(N), \le)$ satisfies the conditions of Zorn's lemma so that it has a maximal element $(\tilde{N}, \tilde{g})$. We want to show that $\tilde{N} = N$ and assume the contrary so that there existed an element $n \in N \setminus \tilde{N}$. Check that the set

$$A := \{r \in R \mid rn \in \tilde{N}\}$$

is an ideal of R. By our assumption, the morphism $A \to \tilde{N} \to I, r \mapsto rn \mapsto \tilde{g}(rn)$ extended to a morphism $\varphi\colon R \to I$. By design, we obtained an extension $\tilde{N} + R \cdot \{n\} \to I, (x, rn) \mapsto \tilde{g}(x) + r\varphi(1)$ as indicated in the diagram

```tikzcd
M \arrow[r, hook] \arrow[d, "f"'] & \tilde{N} \arrow[r, hook] \arrow[dl, "\tilde{g}"] & \tilde{N} \oplus R \cdot \{n\} \arrow[r, hook] \arrow[dll, dotted] & N \\
I. & & &
```

which contradicted the maximality of $(\tilde{N}, \tilde{g})$. Hence $\tilde{N} = N$ which was to show. ☐

Definition 5.2. An abelian group A is called **divisible** iff for every $n \in \mathbf{Z}$ the multiplication morphism $A \xrightarrow{\cdot n} A, a \mapsto na$, is surjective.

Example 5.3.
(i) Any Q-vector space is divisible.
(ii) Any finitely generated abelian group A is not divisible. In fact, $A \cong \mathbf{Z}^n \oplus \bigoplus_{i=1}^k \mathbf{Z}/m_i$ for suitable integers $n \in \mathbf{N}$ and $m_i \in \mathbf{N}_{\ge 1}$.
(iii) Every quotient of a divisible abelian group is divisible. For instance, $\mathbf{Q}/\mathbf{Z}$ is divisible.

Proposition 5.4. *An abelian group is injective if and only if it is divisible.*

Proof. Exercise! Hint: use Baer's criterion. ☐

Theorem 5.5. *The category* Ab *of abelian groups has enough injective objects.*

Proof. Let A be an abelian group. We want to construct a monomoprphism into an injective abelian group. Applying twice the endofunctor

$$(-)^\vee : \mathrm{Ab} \longrightarrow \mathrm{Ab}, \quad X \mapsto X^\vee := \mathrm{Hom}(X, \mathbf{Q}/\mathbf{Z}).$$

to A we get a morphism $\varphi : A \to (A^\vee)^\vee, a \mapsto [f \mapsto f(a)]$. Check that φ is a monomorphism.[12] Choose an epimorphism $f : \bigoplus_J \mathbf{Z} \twoheadrightarrow A^\vee$, e.g. can take $J = A^\vee$. This yields a morphism

$$A \overset{\varphi}{\hookrightarrow} (A^\vee)^\vee \overset{\text{def}}{=} \mathrm{Hom}(A^\vee, \mathbf{Q}/\mathbf{Z}) \xrightarrow{f^*} \mathrm{Hom}(\bigoplus_J \mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \cong \prod_J \mathrm{Hom}(\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \cong \prod_J \mathbf{Q}/\mathbf{Z}$$

which is a momomorphism since f^* is a monomorphism since f is an epimorphism (Lemma 3.4). Since $\mathbf{Q}/\mathbf{Z}$ is injective, the functor $\mathrm{Hom}(-, \mathbf{Q}/\mathbf{Z})$ is exact, hence the functor $\prod_J \mathrm{Hom}(-, \mathbf{Q}/\mathbf{Z}) \cong \mathrm{Hom}(-, \prod_J \mathbf{Q}/\mathbf{Z})$ is exact, hence $\prod_J \mathbf{Q}/\mathbf{Z}$ is an injective abelian group. □

Since the category Ab has enough injective objects, there are right-derived functors associated with any left-exact covariant functor. As an example, let us compute some concrete Ext-groups (cf. Definition 4.19).

Lemma 5.6. *Let A and B be abelian groups. Then* $\mathrm{Ext}^n_{\mathbf{Z}}(A, B) := \mathrm{Ext}^n_{\mathrm{Ab}}(A, B) = 0$ *for* $n \geq 2$

Proof. Choose a monomorphism $B \hookrightarrow I^0$ into an injective abelian group I^0. Then the quotient $I^1 := I^0/B$ is divisible, i.e. injective. Hence $0 \to I^0 \to I^1 \to 0$ is an injective resolution of B. By definition, $\mathrm{Ext}^n_{\mathbf{Z}}(A, B)$ is the n-th cohomology group of the complex

$$0 \longrightarrow \mathrm{Hom}(A, I^0) \longrightarrow \mathrm{Hom}(A, I^1) \longrightarrow 0$$

where $\mathrm{Hom}(A, I^0)$ sits in degree zero. □

Example 5.7. As a special case of Lemma 5.6, for an abelian group A we have that

$$\mathrm{Ext}^n_{\mathbf{Z}}(A, \mathbf{Z}) = \begin{cases} \mathrm{Hom}(A, \mathbf{Z}) & (n = 0), \\ \mathrm{coker}\big(\mathrm{Hom}(A, \mathbf{Q}) \to \mathrm{Hom}(A, \mathbf{Q}/\mathbf{Z})\big) & (n = 1), \\ 0 & (n \geq 2). \end{cases}$$

If A is a torsion group (i.e. every element has finite order), then $\mathrm{Hom}(A, \mathbf{Q}) = 0$ so that $\mathrm{Ext}^1_{\mathbf{Z}}(A, \mathbf{Z}) = \mathrm{Hom}(A, \mathbf{Q}/\mathbf{Z})$.

6. Projective objects and left-derived functors

In this section let $\mathcal{A}$ be an abelian category.

In Remark 4.11 it was stated that one defines also left-derived functors associated with right-exact covariant functors by using projective resolutions instead of injective resolutions. This will be done in this section.

Lemma 6.1 (cf. Lemma 3.4)**.** *For every object X of $\mathcal{A}$ the functor*

$$\mathrm{Hom}_{\mathcal{A}}(X, -) : \mathcal{A} \longrightarrow \mathrm{Ab}, \quad A \mapsto \mathrm{Hom}_{\mathcal{A}}(X, A)$$

is left-exact.

Proof. This is Exercise 3a) on Exercise Sheet 5. □

Definition 6.2 (cf. Definition 3.5)**.** An object P in $\mathcal{A}$ is called **projective** iff the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact.

[12] This is the point where we need $\mathbf{Q}/\mathbf{Z}$ instead of an arbitrary injective abelian group.

Lemma 6.3 (cf. Lemma 3.6). *An object P in $\mathcal{A}$ is projective if and only if for every epimorphism $f: X \to Y$ in $\mathcal{A}$ the induced morphism $f_* = \mathrm{Hom}_{\mathcal{A}}(P,f): \mathrm{Hom}_{\mathcal{A}}(P,X) \to \mathrm{Hom}_{\mathcal{A}}(P,Y)$ is surjective.*

Proof. This is Exercise 3b) on Exercise Sheet 5. □

Lemma 6.4 (cf. Lemma 3.7). *If P is a projective object of $\mathcal{A}$, then every short exact sequence of the form $0 \to X \xrightarrow{f} Y \xrightarrow{g} P \to 0$ splits (cf. Exercise 5 on Exercise Sheet 5).*

Proof. Exercise. □

Lemma 6.5 (cf. Lemma 3.8). *For two objects P and Q of $\mathcal{A}$, the coproduct $P \oplus Q$ is projective if and only if both P and Q are projective.*

Proof. There is an isomorhism of functors $\mathrm{Hom}_{\mathcal{A}}(P \oplus Q, -) \cong \mathrm{Hom}_{\mathcal{A}}(P,-) \oplus \mathrm{Hom}_{\mathcal{A}}(Q,-)$ and the claim follows. □

Lemma 6.6 (cf. Lemma 3.9). *Let $0 \to P' \to P \to P'' \to 0$ be an exact sequence in $\mathcal{A}$ such that P and P'' both are projective. Then also P' is projective.*

Proof. By Lemma 6.4 we have $P \cong P' \oplus P''$ since P'' is projective. By Lemma 6.5 get the claim since P is projective. □

Lemma 6.7. *Let R be a ring. Then an R-module P is projecitve if and only if it is a direct summand of a free R-module.*

Proof. Every free module is projective by Exercise 3c) on Exercise Sheet 5. Hence, if P is a direct summand of a free module, it is projective by Lemma 6.5. For the other direction let P be projective. Choose an epimorphism $p: F := \bigoplus_J \twoheadrightarrow P$ from a free R-module F onto P (e.g. can take $J = P$). Then the exact sequence $0 \to \ker(p) \to F \xrightarrow{p} P \to 0$ splits by Lemma 6.4 so that $F \cong \ker(p) \oplus P$ which was to show. □

Definition 6.8 (cf. Definition 4.1). We say that $\mathcal{A}$ **has enough projective objects** iff for every object A there exists a projective object P and an epimorphism $P \twoheadrightarrow A$.

Proposition 6.9 (cf. Proposition 4.2). *Let R be a ring. Then the category* $\mathrm{Mod}(R)$ *of R-modules has enough projective objects.*

Proof. For an R-module M, the morphism $P := \bigoplus_{m \in M} R \to M, (a_m)_{m \in M} \mapsto \sum_{m \in M} a_m \cdot m$ does the job. □

Remark 6.10. For a ringed space $(X, \mathcal{O}_X)$ the category of $\mathcal{O}_X$-modules need not to have enough projective objects, i.e. the corresponding statement to Theorem 4.3 is not true, see Proposition 8.4.[13]

Definition 6.11 (cf. Definition 4.6). Let A be an object of $\mathcal{A}$. A **projective resolution** of A in $\mathcal{A}$ is an exact sequence $\ldots \to P^{-2} \to P^{-1} \to P^0$ in $\mathcal{A}$ where all obejcts P^{-n} are projective in $\mathcal{A}$ together with a morphism $P^0 \to A$ which is a cokernel of the morphism $P^{-1} \to P^0$. Hence we have an exact sequence

$$\ldots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow A \longrightarrow 0.$$

Proposition 6.12 (cf. Proposition 4.7). *Assume that $\mathcal{A}$ has enough projective objects. Then every object in $\mathcal{A}$ admits a projective resolution.*

Proof. Omitted. □

[13]This implies that the content of this lecture is more than bare abstract nonsense.

Construction 6.13 (cf. Construction 4.8)**.** Assume that $\mathcal{A}$ has enough projectives objects and let $F\colon\mathcal{A}\to\mathcal{B}$ be a right-exact additive functor into an abelian category $\mathcal{B}$. For an object A in $\mathcal{A}$ we choose a projective resolution $\ldots \xrightarrow{d_P^{-3}} P^{-2} \xrightarrow{d_P^{-2}} P^{-1} \xrightarrow{d_P^{-1}} P^0 \longrightarrow A \longrightarrow 0$. Forgetting A and applying the functor F the induced sequence

$$\ldots \xrightarrow{F(d_P^{-3})} F(P^{-2}) \xrightarrow{F(d_P^{-2})} F(P^{-1}) \xrightarrow{F(d_P^{-1})} F(P^0) \xrightarrow{F(d_P^0)} 0.$$

may not be exact anymore, but it is still a complex, i.e. $F(d_P^{-n+1})\circ F(d_I^{-n}) = 0$ for every $n \geq 0$. Thus we can define for every $n \in \mathbf{N}$ the cohomology group

$$\mathrm{L}^n_{P^\bullet} F(A) := \mathrm{H}^{-n}(F(P^\bullet)) = \frac{\ker(F(d_P^{-n}))}{\operatorname{im}(F(d_P^{-n-1}))}.$$

Theorem 6.14 (Existence and properties of left-derived functors, cf. Theorem 4.9)**.** *Assume that $\mathcal{A}$ has enough projective objects and let $F\colon\mathcal{A}\to\mathcal{B}$ a right-exact additive functor into another abelian category $\mathcal{B}$.*

(1) (Uniqueness) *The definition of $\mathrm{L}^n_{P^\bullet} F(A)$ is independent of the choice of the projective resolution so that we can just write $\mathrm{L}^n F(A)$.*

(2) (Functoriality) *For every $n \in \mathbf{N}$ we get an additive functor*

$$\mathrm{L}^n F\colon \mathcal{A}\to\mathcal{B}, \quad A \mapsto \mathrm{L}^n F(A).$$

(3) $F \cong \mathrm{L}^0 F$.

(4) (Long exact sequence) *For every short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

and for every $n \in \mathbf{N}$ there exists a canonical homomorphism

$$\partial^n\colon \mathrm{L}^{n-1} F(C) \to \mathrm{L}^n F(A)$$

and a long exact sequence

$$\ldots \longrightarrow \mathrm{L}^n F(A) \xrightarrow{\mathrm{L}^n F(f)} \mathrm{L}^n F(B) \xrightarrow{\mathrm{L}^n F(g)} \mathrm{L}^n F(C) \xrightarrow{\partial^n} \mathrm{L}^{n-1} F(A) \longrightarrow \ldots$$

(5) (Functoriality for LES) *For every commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{A}$ with exact rows and every $n \in \mathbf{N}$ we get a commuative diagram

$$\begin{tikzcd} \mathrm{L}^n F(A) \arrow[r] \arrow[d] & \mathrm{L}^n F(B) \arrow[r] \arrow[d] & \mathrm{L}^n F(C) \arrow[r, "\partial^n"] \arrow[d] & \mathrm{L}^{n-1} F(A) \arrow[d] \\ \mathrm{L}^n F(A') \arrow[r] & \mathrm{L}^n F(B') \arrow[r] & \mathrm{L}^n F(C') \arrow[r, "\partial^n"] & \mathrm{L}^{n-1} F(A'). \end{tikzcd}$$

Proof. The proof is dual to the proof of Theorem 4.9. □

Definition 6.15. Let R be a ring. Then the category $\mathrm{Mod}(R)$ of R-modules has enough projective objects (Proposition 6.9) and for every R-module X, the endofunctor $X \otimes_R (-)\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ is right-exact. Its derived functors

$$\mathrm{Tor}_n^R(X,-) := \mathrm{L}^n\big(X \otimes_R (-)\big)\colon \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(R), A \mapsto \mathrm{Tor}_n^R(X,A)$$

are called the **Tor-functors**.

Theorem 6.16. *Let R be a ring. For $n \geq 0$ and any two R-modules A and B we have a canonical and funcotorial isomorphism* $\operatorname{Tor}_n^R(A,B) \cong \operatorname{Tor}_n^R(B,A)$.

Proof. Outsources, see [Wei94, Thm. 2.7.2]. □

Example 6.17. For $m \in \mathbf{N}$ the exact sequence $0 \to \mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z} \longrightarrow \mathbf{Z}/m \to 0$ is a projective resolution. Hence we have for any abelian group A that

$$\operatorname{Tor}_n^{\mathbf{Z}}(A,\mathbf{Z}/m) = \begin{cases} A \otimes_{\mathbf{Z}} \mathbf{Z}/m = A/mA & (n=0), \\ \ker\left(A \otimes_{\mathbf{Z}} \mathbf{Z} \overset{\mathrm{id}_A \otimes (\cdot m)}{\to} A \otimes_{\mathbf{Z}} \mathbf{Z}\right) = \ker(A \overset{\cdot m}{\to} A) = A[n] & (n=1), \\ 0 & (n \geq 2), \end{cases}$$

where $A[m] := \{a \in A \mid m \cdot a = 0\}$ is the m-torsion part of A.

Lemma 6.18. *Let A and B be abelian groups. Then $\operatorname{Tor}_1^{\mathbf{Z}}(A,B)$ is a torsion group (i.e. every element has finite order) and $\operatorname{Tor}_n^{\mathbf{Z}}(A,B) = 0$ for $n \geq 2$.*

Proof. Since every abelian group is the filtered colimit of its finitely generated subgroups and since filtered colimits of torsion groups are torsion, we may assume that B is finitely generated, say $B \cong \mathbf{Z}^r \oplus \bigoplus_{i=1}^k \mathbf{Z}/p_i$ for integers $r \in \mathbf{N}$ and $m_i \in \mathbf{N}_{\geq 1}$. Since Tor commutes with coproducts (exercise) we get that

$$\operatorname{Tor}_n^{\mathbf{Z}}(A,B) \cong \underbrace{\operatorname{Tor}_n^{\mathbf{Z}}(A,\mathbf{Z}^r)}_{=0} \oplus \bigoplus_{i=1}^{k} \operatorname{Tor}_n^{\mathbf{Z}}(A,\mathbf{Z}/m_i)$$

which implies the claim by Example 6.17. □

Proposition 6.19. *An abelian group A is torsion-free if and only if $\operatorname{Tor}_1^{\mathbf{Z}}(B,A) = 0$ for every abelian group B.*

Proof. Again, we may assume that A is finitely generated. If A is torsion-free, then $A \cong \mathbf{Z}^r$ is projective. The other way, if A has an element of order n, then $\operatorname{Tor}_1^{\mathbf{Z}}(\mathbf{Z}/n,A) \cong \operatorname{Tor}_1^{\mathbf{Z}}(A,\mathbf{Z}/n) = A[n] \neq 0$ by Theorem 6.16. □

7. Left-/right-derived co-/contra-variant functors (summary)

We defined right-derived functors associated with a left-exact covariant functor (Definition 4.10) and proved some key properties of them (Theorem 4.9). In the same spirit one can proceed for any left-/right-exact co-/contra-variant functor to get respective right-/left-derived functors:

- For a left-exact covariant functor $F\colon \mathcal{A} \to \mathcal{B}$ one defines its right-derived functors $\left(\mathrm{R}^n F\colon \mathcal{A} \to \mathcal{B}\right)_{n \in \mathbf{N}}$ in terms of injective resolutions (Theorem 4.9).
- For a left-exact contravariant functor $G\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ one defines its right-derived functors $\left(\mathrm{R}^n G\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}\right)_{n \in \mathbf{N}}$ in terms of projective resolutions.
- For a right-exact covariant functor $F\colon \mathcal{A} \to \mathcal{B}$ one defines its left-derived functors $\left(\mathrm{L}^n F\colon \mathcal{A} \to \mathcal{B}\right)_{n \in \mathbf{N}}$ in terms of projective resolutions (Theorem 6.14).
- For a right-exact contravariant functor $G\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ one defines its left-derived functors $\left(\mathrm{L}^n G\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}\right)_{n \in \mathbf{N}}$ in terms of injective resolutions.

To display this in a tabular, let A be an object of $\mathcal{A}$, let $A \hookrightarrow I^\bullet$ be an injective resolution, and let $P^\bullet \twoheadrightarrow A$ be a projective resolution.

	covariant $F\colon \mathcal{A} \to \mathcal{B}$	contravariant $G\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$
left-exact	$\mathrm{R}^n F(A) := \mathrm{H}^n(F(I^\bullet))$	$\mathrm{R}^n G(A) := \mathrm{H}^n(G(P^\bullet))$
right-exact	$\mathrm{L}^n G(A) := \mathrm{H}^n(G(P^\bullet))$	$\mathrm{L}^n G(A) := \mathrm{H}^n(G(I^\bullet))$

Definition 7.1 (cf. Definition 4.19)**.** For every object Y in an abelian category $\mathcal{A}$ the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-,Y)$ is left-exact (Lemma 3.4). For $n \in \mathbf{N}$ call its n-th right-derived functor

$$\tilde{\mathrm{Ext}}^n_{\mathcal{A}}(-,Y) := \mathrm{R}^n\,\mathrm{Hom}_{\mathcal{A}}(-,Y)\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathrm{Ab}, \quad A \mapsto \tilde{\mathrm{Ext}}^n_{\mathcal{A}}(Y,A).$$

In order to compute $\tilde{\mathrm{Ext}}^n_{\mathcal{A}}(A,Y)$ for an object A of $\mathcal{A}$, one chooses a projective resolution $P^\bullet \twoheadrightarrow A$; then

$$\tilde{\mathrm{Ext}}^n_{\mathcal{A}}(A,Y) = \mathrm{H}^n(\mathrm{Hom}_{\mathcal{A}}(P^\bullet,Y)) = \frac{\ker\big(\mathrm{Hom}_{\mathcal{A}}(P^{-n},Y) \xrightarrow{(d_P^{-n})^*} \mathrm{Hom}_{\mathcal{A}}(P^{-n+1},Y)\big)}{\mathrm{im}\big(\mathrm{Hom}_{\mathcal{A}}(P^{-n-1},Y) \xrightarrow{(d_P^{-n-1})^*)} \mathrm{Hom}_{\mathcal{A}}(P^{n},Y)\big)}.$$

where $\mathrm{Hom}_{\mathcal{A}}(P^{-n},Y)$ sits in degree $+n$ (taking into account that $\mathrm{Hom}_{\mathcal{A}}(-,Y)$ reverses arrows).

Theorem 7.2. *Let $\mathcal{A}$ be an abelian category which has both enough injective and enough projecitve objects. Then, for any two objects A and B in $\mathcal{A}$, there exists a canonical isomorphism*

$$\mathrm{Ext}^*_{\mathcal{A}}(A,B) \cong \tilde{\mathrm{Ext}}^*_{\mathcal{A}}(A,B)$$

which is functorial in A and in B and also compatible with the boundary morphisms. In particular, for any injective resolution $B \hookrightarrow I^\bullet$ and any projective resolution $P^\bullet \twoheadrightarrow A$ there exists for every $n \in \mathbf{N}$ a canonical isomorphism

$$\mathrm{H}^n(\mathrm{Hom}_{\mathcal{A}}(A,I^\bullet)) \cong \mathrm{H}^{-n}(\mathrm{Hom}_{\mathcal{A}}(P^\bullet,B)).$$

The assertion also implies that we get additive functors

$$\mathrm{Ext}^n(-,-)\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathrm{Ab}.$$

Proof. This is not done in the lecture, for a proof consider [Wei94, Thm. 2.7.6]. □

8. Enough projective sheaves

Let R be a ring. For any topological space X the category $\mathrm{Sh}(X,R)$ has enough injective objects (Corollary 4.4). This is deduced from the fact that the category $\mathrm{Mod}(R)$ has enough injective objects (Proposition 4.2). Since the category $\mathrm{Mod}(R)$ also has enough projective objects (Proposition 6.9), one might ask whether one can also deduce that the category $\mathrm{Sh}(X,R)$ has enough projective objects. In general, this is not true. But it is true if the space X is finitely generated, as we shall see.

Definition 8.1. A topological space X is called **finitely generated** iff every point $x \in X$ has a smallest open neighbourhood.

Example 8.2. The real numbers $\mathbf{R}$ equipped with the euclidean topology is not finitely generated.

Proposition 8.3. *For a topological space X the following conditions are equivalent:*

- *(i) X is finitely generated.*
- *(ii) Any intersection of open subests of X is an open subset of X.*
- *(iii) Any union of closed subsets of X is a closed subset of X.*

Proof. Clearly, (iii)⇔(ii)⇒(i). We assume (i) and denote by U_x the smallest open neighbourhood of x. Let $V := \bigcap_{j\in J} V_j$ be an intersection of open subsets of X. If $x \in V$, then $x \in V_j$ for all $j \in J$. Hence $U_x \subset V$ so that the latter is open. □

Proposition 8.4. *Let X be a topological space. Then the category $\mathrm{Sh}(X,R)$ of sheaves of R-modules on X has enough projective objects if and only if X is finitely generated.*

Proof. Assume that X is finitely generated. Let $x \in X$ and denote by U_x is smallest open neighbourhood. Let $j_x : U_x \to X$ be the inclusion map.

Claim: The sheaf $(j_x)_! \underline{R}$ is projective.

Let $f : F \to G$ be an epimorphism and $g : (j_x)_! \underline{R} \to G$ be any morphism. We have to solve the lifting problem

$$\begin{tikzcd} & F \arrow[d, "f"] \\ (j_x)_! \underline{R} \arrow[r, "g"'] \arrow[ur, dotted, "\exists h?"] & G \end{tikzcd}$$

of sheaves on X. By the adjunction $((j_x)_!, j_x^{-1})$ this is equivalent to the lifting problem

$$\begin{tikzcd} & j_x^{-1} F \arrow[d, "j_x^{-1} f"] \\ \underline{R} \arrow[r, "j_x^{-1} g"'] \arrow[ur, dotted, "\exists h^{\mathrm{adj}}?"] & j^{-1} G \end{tikzcd}$$

of sheaves on U_x. Since the functor $j_x^{-1} : \mathrm{Sh}(X, R) \to \mathrm{Sh}(U_x, R)$ is exact, the morphism $j_x^{-1} f$ is an epimorphism. Hence the morphism $j_x^{-1}(U_x) : j_x^{-1} F(U_x) \to j_x^{-1} G(U_x)$ is an epimorphism as U_x is the only open neighbourhood of x in U_x. Thus we get the desired lift $h^{\mathrm{adj}} : \underline{R} \to j_x^{-1} F$ which is determined by the morphism $h^{\mathrm{adj}}(U_x) : \underline{R}(U_x) = R \to j_x^{-1}(U_x)$ mapping 1 to some preimage of $j_x^{-1} g(U_x)(1)$.

Now let F be a sheaf of R-modules on X. We want to construct an epimorphism $P \to F$ from a projective sheaf of R-modules P. For every $x \in X$ choose an epimorphism of R-modules $f_x : \bigoplus_{J_x} R \to F_x$. This yields a morphism $f_x : \bigoplus_{J_x} \underline{R} \to j^{-1} F$ and hence a morphism $f_x^{\mathrm{adj}} : P_x := \bigoplus_{J_x} j_x^{-1} \underline{R} \to F$ which both are epimorphisms on the stalks at x. Hence the induced morphism $P := \bigoplus_{x \in X} P_x \to F$ is an epimorphism from the projective sheaf P to F.

For the other direction assume that X is *not* finitely generated. Then there exists a point $x \in X$ which does not have a smallest open neighbourhood. Let $\iota : \{x\} \hookrightarrow X$ be the inclusion map.

Claim: The sheaf $\iota_* \underline{R}$ does not admit an epimorphism from a projective sheaf.

Let U be a connected neighbourhood of x. By assumption we find a strictly smaller neigbouhood $x \in V \subsetneq U$. Let $j : V \hookrightarrow X$ be the inclusion map and consider the sheaf $j_! \underline{R}$; its stalks are as follows [Stacks, Tag 00A6]:

$$(j_! \underline{R})_y = \begin{cases} R & (y \in V), \\ 0 & (y \notin V). \end{cases}$$

There is a canonical epimorphism $\mathrm{can} : j_! \underline{R} \to i_* \underline{R}$. Assume that there existed an epimorphism $p : P \to i_* \underline{R}$ with P being projective. Then we would get a lift $h : P \to j_! \underline{R}$.

$$\begin{tikzcd} & j_! \underline{R} \arrow[d, "\mathrm{can}"] \\ P \arrow[r, "p"'] \arrow[ur, dotted, "h"] & i_* \underline{R} \end{tikzcd} \qquad \begin{tikzcd} & j_! \underline{R}(U) = 0 \arrow[d, "\mathrm{can}(U)"] \\ P(U) \arrow[r, "p(U)"'] \arrow[ur, "h(U)"] & i_* \underline{R}(U) = R \end{tikzcd}$$

Since $U \nsubseteq V$ and U is connected, $j_! \underline{R}(U) = 0$ (exercise), hence the morphism $p(U) : P(U) \to i_* \underline{R}(U) = R$ would be zero. Now you can check that the induced morphism $p_x : P_x \to (i_* \underline{R})_x = R$ on stalks would be zero as well which would be a contradiction to p being an epimorphism. $\square$

Examining the proof of Propositon 8.4 one gets:

Porism 8.5. *Let $(X,\mathcal{O}_X)$ be a ringed space. If the space X is finitely generated, then the category $\mathrm{Mod}(\mathcal{O}_X)$ of $\mathcal{O}_X$-modules has enough projective objects. Furthermore, if $\mathrm{Mod}(\mathcal{O}_X)$ has enough projective objects and $\mathcal{O}_{X,x} \neq 0$ for all $x \in X$, then X is finitely generated.*

In the proof of Proposition 8.4 we used the following helpful lemma:

Lemma 8.6. *Let $L\colon \mathcal{A} \rightleftarrows \mathcal{B}\colon R$ be an adjunction between abelian categories.*

(i) If L left-exact, then R preserves injective objects.
(ii) If R is right-exact, then L preserves projective objects.

Proof. Exercise. □

9. Sheaf cohomology

In this section let X be a topological space and let R be a ring. All sheaves in this section are sheaves of R-modules.

Proposition 9.1. *The global sections functor*

$$\Gamma(X,-)\colon \mathrm{Sh}(X,R) \longrightarrow \mathrm{Mod}(R), \quad F \mapsto \Gamma(X,F) = F(X),$$

is left-exact.

Proof. This was essentially done in the first half of the course. One can compose $\Gamma(X,-)$ as

$$\mathrm{Sh}(X,R) \xrightarrow{\mathrm{incl}} \mathrm{PSh}(X,R) \xrightarrow{\Gamma^{\mathrm{pre}}(X,-)} \mathrm{Mod}(R)$$

where the inclusion functor is left-exact and the presheaf global section functor is exact. □

Definition 9.2. For $n \in \mathbf{N}$, the **n-th cohomology functor** $\mathrm{H}^n(X,-)$ is defined to be the n-derived functor (Definition 4.10) of the left-exact global sections functor:

$$\mathrm{H}^n(X,-) := \mathrm{R}^n\Gamma(X,-)\colon \mathrm{Sh}(X,R) \longrightarrow \mathrm{Mod}(R),$$
$$F \mapsto \mathrm{H}^n(X,F) := \big(\mathrm{R}^n\Gamma(X,-)\big)(F).$$

The properties of right-derived functors immediately yield the first part of Theorem 1.3:

Theorem 9.3 (Existence and properties of sheaf cohomology, part I)**.** *Let R be a ring. For every sheaf of R-modules F on a topological space X and for every natural number $n \geq 0$ there exists an R-module $\mathrm{H}^n(X,F)$ such that the following holds:*

(1) (Functoriality for the sheaf) *The assignment of objects $\mathrm{Sh}(X,R) \ni F \mapsto \mathrm{H}^n(X,F) \in \mathrm{Mod}(R)$ can be extended to a functor $\mathrm{H}^n(X,-)\colon \mathrm{Sh}(X,R) \to \mathrm{Mod}(R)$.*
(2) For $n = 0$, the functor $\mathrm{H}^0(X,-)$ is canonically isomorphic to the global sections functor $\Gamma(X,-)$.
(3) (Long exact cohomology sequence) *For every short exact sequence $0 \to F \to G \to H \to 0$ of sheaves of R-modules on X there exist a canonical homomorphisms $\partial\colon \mathrm{H}^n(X,H) \to \mathrm{H}^{n+1}(X,F)$ such the the induced sequence*

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,F) \arrow[r] & \Gamma(X,G) \arrow[r] & \Gamma(X,H) \arrow[dll] \\
& \mathrm{H}^1(X,F) \arrow[r] & \mathrm{H}^1(X,G) \arrow[r] & \mathrm{H}^1(X,H) \arrow[dll] \\
& \mathrm{H}^2(X,F) \arrow[r] & \mathrm{H}^2(X,G) \arrow[r] & \ldots\ldots
\end{tikzcd}$$

of R-modules is exact.[14]

(4) *For every commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F' \arrow[r] & G' \arrow[r] & H' \arrow[r] & 0 \end{tikzcd}$$

of sheaves of R-modules on X with exact rows we get a commuative diagram

$$\begin{tikzcd} \mathrm{H}^n(X,F) \arrow[r] \arrow[d] & \mathrm{H}^n(X,G) \arrow[r] \arrow[d] & \mathrm{H}^n(X,H) \arrow[r, "\partial"] \arrow[d] & \mathrm{H}^{n+1}(X,F) \arrow[d] \\ \mathrm{H}^n(X,F') \arrow[r] & \mathrm{H}^n(X,G') \arrow[r] & \mathrm{H}^n(X,H') \arrow[r, "\partial"] & \mathrm{H}^{n+1}(X,F'). \end{tikzcd}$$

Proof. This is Theorem 6.14 applied to the functor $\Gamma(X,-)$. □

We have seen that one can also use acyclic resolutions to compute derived functors (Proposition 4.23). For the global sections functor $\Gamma(X,-)$ we will see that flasque sheaves are acyclic.

Definition 9.4. A sheaf F on X is called **flasque** iff for every inclusion $V \subseteq U$ of open subsets of X the restriction morphism $F(U) \to F(V)$ is surjective.

Examples 9.5. (i) The sheaf of holomorphic function $\mathcal{O}_{\mathbf{C}}$ on the complex numbers is not flasque, whereas the sheaf $\mathcal{M}_{\mathbf{C}}$ of meromorphic functions is flasque.

(ii) Let X be an irreducible topological space. Then any F constant sheaf on X is flasque.

Proof. This is an exercise. □

A class of examples of flasque sheaves are the injective ones:

Proposition 9.6. *Let $(X,\mathcal{O}_X)$ be a ringed space. Every injective $\mathcal{O}_X$-module is flasque.*

Proof. Let $V \subseteq U$ be open subsets of X and denote by $v: V \hookrightarrow X$ and $u: U \hookrightarrow X$ their inclusions. Then the canonical morphism $v_! v^{-1}\mathcal{O}_X \to u_! u^{-1}\mathcal{O}_X$ is a monomorphism (check this on stalks). For an injective $\mathcal{O}_X$-module I the induced morphism

$$\mathrm{Hom}(u_! u^{-1}\mathcal{O}_X, I) \to \mathrm{Hom}(v_! v^{-1}\mathcal{O}_X, I)$$

is an epimorphism. Using the adjunction $(u_!, u^{-1})$ we get

$$\mathrm{Hom}(u_! u^{-1}\mathcal{O}_X, I) \cong \mathrm{Hom}(\mathcal{O}_U, u^{-1}I) \cong u^{-1}I(U) = I(U)$$

and analogously $\mathrm{Hom}(v_! v^{-1}\mathcal{O}_X, I) \cong I(V)$ which implies the claim. □

One can consider the notion of a flasque sheaf a a generalisation of the notion of an injective sheaf. As we shall see later in Theorem 9.10, all flasque sheaves are $\Gamma(X,-)$-acyclic. The following statement for injective sheaves is true already by definition.

Lemma 9.7 (cf. Definition 3.5). *Let $0 \to F \to G \to H \to 0$ be a short exact sequence of sheaves such that F is flasque. Then for every open subset U of X the induced sequence of sections*

$$0 \to F(U) \to G(U) \to H(U) \to 0$$

is exact.

[14] The conditions (2) and (3) say that the family $(\mathrm{H}^n(X,-))_{n\geq 0}$ is a δ-functor, cf. Grothendieck's Tohoku paper [Gro57].

Proof. We have that $H = G/F$. As the global sections functor is left-exact, for every open subset U of X the sequence of R-modules

$$0 \to F(U) \to G(U) \to G(U)/F(U) \to 0$$

is exact.

We shall show that the presheaf defined by $U \mapsto G(U)/F(U)$ is already a sheaf. In this case we get $G(U)/F(U) = (G/F)(U) = H(U)$ which implies the theorem.

Let $(U_i)_{i\in I}$ be an open cover of U. For an open subset V of X and a section $s \in G(V)$ we denote by $\bar{s}$ the associated element in $G(V)/F(V)$.

Locality. Let $s \in G(U)$ such that $\bar{s}|_{U_i} = 0$ for all $i \in I$. Hence $s|_{U_i} \in F(U_i)$ and, as F is flasque, we find for every $i \in I$ a section $s'_i \in F(X)$ such that $s'_i|_{U_i} = s|_{U_i}$ and for all $i,j \in I$ we have that

$$s'_i|_{U_i\cap U_j} = s|_{U_i\cap U_j} = s'_j|_{U_i\cap U_j}.$$

Since F is a sheaf, the sections $s'_i|_{U_i}$ glue to a section $s' \in F(X)$ such that $s'|_{U_i} = s'_i|_{U_i} = s|_{U_i}$. By locality for F we get $s \in F(X)$ and $\bar{s} = 0$.

Glueing. For every $i \in I$ let $\bar{s}_i \in G(U_i)/F(U_i)$ such that

$$\bar{s}_i|_{U_i\cap U_j} = \bar{s}_j|_{U_i\cap U_j}$$

for all $i,j \in I$. Then there exists preimages $(s_i \in G(U_i))_{i\in I}$ and there exist $s_{ij} \in F(U_i \cap U_j)$ such that

$$s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j} = s_{ij}.$$

Consider the set

$$M := \{(V,s) \mid V \subseteq U \text{ open, } s \in G(V) \text{ such that } \bar{s}|_{U_i} = \bar{s_i} \text{ in } G(V)/F(V)\}$$

equipped with the order

$$(V,s) \le (W,t) :\Leftrightarrow \big(V \subseteq W \;\&\; s = t|_V\big).$$

We check the assumptions in Zorn's lemma: Since the (U_i, s_i) are elements of M, we have that $M \neq \varnothing$. Now let $T := \{(V_j,t_j) | j \in J\}$ be a totally ordered subset of M. We set $V_J := \bigcup_{j\in J} V_j$. Since G is a sheaf, we find a section $t \in G(V_J)$ such that $t|_{V_j} = t_j$. Hence (V_J, t) is an upper bound of T in M.

By Zorn's lemma, there exists a maximal element (V,s) in M.

Assume that $V \neq U$. Then there existed an $i \in I$ such that $U_i \nsubseteq V$ and a section $t \in F(U_i \cap V)$ such that $s|_{U_i\cap V} = s_i|_{U_i\cap V} + t$. Since F is flasque, there existed a preimage $t_i \in F(U_i)$ such that $t_i|_{U_i\cap U} = t$. Then $s_i + t_i$ was a section in $G(U_i)$ and we could glue s and $s_i + t_i$ together to a section $s' \in G(U_i \cup V)$ and would get $(U_i \cup V, s') \in M$ which is a contradiction. Hence $V = U$ and its residue class $\bar{s}$ in $G(U)/F(U)$ is the global section we were looking for. □

Corollary 9.8 (cf. Lemma 3.9). *Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves on X and assume that F and G are flasque. Then H is flasque as well.*

Proof. Let $V \subseteq U$ be open subsets of X. Since F is flasque we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F(U) \arrow[r] \arrow[d] & G(U) \arrow[r] \arrow[d] & H(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(V) \arrow[r] & G(V) \arrow[r] & H(V) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since G is flasque, the map

$$H(U) = G(U)/F(U) \to G(V)/F(V) = H(V)$$

is surjective and hence H is flasque. □

The following stability property for flasque sheaves is very helpful; the corresponding statement for injective sheaves is not true.

Lemma 9.9. *Let $f\colon X \to Y$ be a continuous map into another topological space Y and let F be a flasque sheaf on X. Then the pushforward sheaf f_*F is flasque as well.*

Proof. Let $V \subseteq U$ be open subset of Y. Then $f^{-1}(V) \subseteq f^{-1}(U)$ are open subsets of X. Since F is flasque, the map

$$(f_*F)(U) = F(f^{-1}(U)) \to F(f^{-1}(V)) = (f_*F)(V)$$

is surjective and hence f_*F is flasque. □

Theorem 9.10. *Every flasque sheaf on X is $\Gamma(X,-)$-acyclic.*

Proof. Let F be a flasque sheaf on X and let I be an injective sheaf such that the sequence

$$0 \to F \to I \to I/F \to 0$$

is exact and hence I/F flasque by Corollary 9.8. Since F is flasque. the sequence

$$0 \to \Gamma(X,F) \to \Gamma(X,I) \to \Gamma(X,I/F) \to 0 \tag{$\heartsuit$}$$

is exact. We get a long exact sequence of cohomology groups

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,F) \arrow[r] & \Gamma(X,I) \arrow[r] & \Gamma(X,I/F) \arrow[dll, "\partial"'] \\
& \mathrm{H}^1(X,F) \arrow[r] & \mathrm{H}^1(X,I) \arrow[r] & \mathrm{H}^1(X,I/F) \arrow[dll, "\partial"'] \\
& \mathrm{H}^2(X,F) \arrow[r] & \mathrm{H}^2(X,I) \arrow[r] & \cdots\cdots
\end{tikzcd}$$

As I is injective, we have $\mathrm{H}^n(X,I) = 0$ and hence the boundary morphism $\partial\colon \mathrm{H}^n(X,I/F) \to \mathrm{H}^{n+1}(X,F)$ is an isomorphism for every $n \geq 1$.

Now we proceed by induction on n. Since the sequence ($\heartsuit$) is exact, we have $\mathrm{H}^1(X,F) = 0$ and the same is true for any flasque sheaf, in particular for I/F. With the isomorphism above we are done. □

Lemma 9.11. *Let F be a sheaf on X and let $n \in \mathbf{N}$.*

(i) Let F be a sheaf on X and let U be an open subset X. Then $\mathrm{H}^n(U,F) = \mathrm{H}^n(U,F|_U)$.
(ii) Let $i\colon Z \hookrightarrow$ be the inclusion of a closed subset and let G be a sheaf on Z. Then $\mathrm{H}^n(X,i_*G) \cong \mathrm{H}^n(Z,G)$

Proof. (i) Let $F \hookrightarrow \boldsymbol{I}^\bullet$ be an injective resolution of sheaves on X. Let $j\colon U \hookrightarrow X$ be the inclusion map so that $F|_U = j^{-1}F$. Since the left-adjoint $j_!$ is exact, the sheaves $j^{-1}I^k$ are injective sheaves on U for every $k \in \mathbf{N}$ (Lemma 8.6). Since j^{-1} is exact, the induced sequence $j^{-1}F \to j^{-1}\boldsymbol{I}^\bullet$ is exact, hence an injective resolution. Thus

$$\mathrm{H}^n(U,F) = \mathrm{H}^n(\boldsymbol{I}^\bullet(U)) = \mathrm{H}^n(j^{-1}\boldsymbol{I}^\bullet(U)) = \mathrm{H}^n(U,j^{-1}F) = \mathrm{H}^n(U,F|_U).$$

(ii) Let $G \hookrightarrow \boldsymbol{J}^\bullet$ be an injective resolution of sheaves on Z. Then for every $k \in \mathbf{N}$ the sheaf i_*J^k is flasque by Lemma 9.9. Since i_* is exact, the induced sequence $i_*G \hookrightarrow i_*\boldsymbol{J}^\bullet$ is an $\Gamma(X,-)$-acyclic resolution, hence it computes $\mathrm{H}^n(X,i_*G)$. Then

$$\mathrm{H}^n(X,i_*G) = \mathrm{H}^n(i_*\boldsymbol{J}^\bullet(X)) = \mathrm{H}^n(\boldsymbol{J}^\bullet(Z)) = \mathrm{H}^n(Z,G).$$

□

Proposition 9.12 (Mayer-Vietoris sequences). *Let F be a sheaf on X.*

(i) Let U and V be two open subsets of X. Then we have an exact sequence

$$\ldots \to \mathrm{H}^n(U \cup V, F) \to \mathrm{H}^n(U,F) \oplus \mathrm{H}^n(V,F) \to \mathrm{H}^n(U \cap V, F) \xrightarrow{\partial} \mathrm{H}^{n+1}(U \cup V, F) \to \ldots$$

(ii) Let Y and Z be two closed subsets of X. Then we have an exact sequence

$$\ldots \to \mathrm{H}^n(Y \cup Z, F|_{Y \cup Z}) \to \mathrm{H}^n(Y, F|_Y) \oplus \mathrm{H}^n(Z, F|_Z) \to \mathrm{H}^n(Y \cap Y, F|_{Y \cap Z}) \xrightarrow{\partial} \mathrm{H}^{n+1}(Y \cup Z, F|_{Y \cup Z}) \to \ldots$$

Proof. (i) Let $F \hookrightarrow I^\bullet$ be an injective resolution of sheaves on X. Using the sheaf condition and injectivity we get an exact sequence of complexes

$$0 \longrightarrow I^\bullet(U \cup V) \xrightarrow{\alpha} I^\bullet(U) \oplus I^\bullet(V) \xrightarrow{\beta} I^\bullet(U \cap V) \longrightarrow 0$$

where $\alpha(s) := (\alpha|_U, \alpha|_V)$ and $\beta(s,t) := s|_{U \cap V} - s|_{U \cap V}$ are induced by the level-wise restriction morphisms. Then the long exact cohomology sequence (Proposition 4.17) and Lemma 9.11 yield the desired long exact sequence.

(ii) For a closed subset $A \subset X$ with inclusion map $i: A \hookrightarrow X$ we write $F_A := i_* i^{-1} F = i_* F|_A$. Check that for two closed subsets $A \subset B$ in X the restriction morphisms induce a morphism $\varrho^B_A: F_B \to F_A$ of sheaves on X. Then the sequence

$$0 \longrightarrow F_{Y \cup Z} \xrightarrow{\alpha} F_Y \oplus F_Z \xrightarrow{\beta} F_{Y \cap Z} \longrightarrow 0$$

is exact where $\alpha = (\varrho^{Y \cup Z}_Y, \varrho^{Y \cup Z}_Z)$ and $\beta = \varrho^Y_{Y \cap Z} - \varrho^Z_{Y \cap Z}$. Then the long exact sheaf cohomology sequence (Theorem 9.3 (3)) and lemma 9.11 yield the desired long exact sequence. □

So far we have considered sheaf cohomology on a fixed space and its subspaces. Now look at the behaviour with respect to continuous maps.

Proposition 9.13 (Functoriality for the space). *Let $f: X \to Y$ be a continuous map of topological spaces and let $n \in \mathbf{N}$.*

(i) For every sheaf G on Y there exists a morphism

$$f^*: \mathrm{H}^n(Y, G) \longrightarrow \mathrm{H}^n(X, f^{-1}G)$$

which is compatible with long exact cohomology sequences.

(ii) For every sheaf F on X there exists a morphism

$$f^*: \mathrm{H}^n(Y, f_*F) \longrightarrow \mathrm{H}^n(X, F).$$

In case that f_* is exact, this is compatible with long exact cohomology sequences.

Proof. (i) Let $G \hookrightarrow I^\bullet$ be an injective resolution of sheaves on Y and let $f^{-1}G \hookrightarrow J^\bullet$ be an injective resolution of sheaves on X. Since the left-adjoint f^{-1} is exact, the right-adjoint f_* preserves injective objects (Lemma 8.6). Thus $f_* J^\bullet$ is a complex of injective sheaves on Y. By Proposition 4.15 the unit morphism $\eta: G \to f_* f^{-1} G$ extends to a morphism $\eta^\bullet: I^\bullet \to f_* J^\bullet$ of complexes of sheaves on Y. Hence we get an induced morphism

$$\mathrm{H}^n(Y, G) = \mathrm{H}^n(I^\bullet(Y)) \to \mathrm{H}^n(f_* J^\bullet(Y)) = \mathrm{H}^n(J^\bullet(X)) = \mathrm{H}^n(X, f^{-1}G).$$

(ii) Let $F \hookrightarrow J^\bullet$ be an injective resolution of sheaves on X and let $f_*F \hookrightarrow I^\bullet$ be an injective resolution of sheaves on Y. The induced sequence $f_* J^\bullet$ is a complex of injective sheaves on Y. By Proposition 4.15, the identity morphism $\mathrm{id}: f_*F \to f_*F$ extens to a morphism $I^\bullet \to f_* J^\bullet$ of sheaves on Y. Hence we get an induced morphism

$$\mathrm{H}^n(Y, f_*F) = \mathrm{H}^n(I^\bullet(Y)) \to \mathrm{H}^n(f_* J^\bullet(Y)) = \mathrm{H}^n(J^\bullet(X)) = \mathrm{H}^n(X, F).$$

The compatibility with long exact cohomology sequences is an exercise. □

Proposition 9.14 (Localisation). *Let $j: U \hookrightarrow X$ be the inclusion of an open subset and let $i: Z := X \setminus U \hookrightarrow X$ be the inclusion of the closed complement. For every sheaf F on X there is an exact sequence*

$$\dots \to \mathrm{H}^n(U, F|_U) \to \mathrm{H}^n(X, F) \to \mathrm{H}^n(Z, F|_Z) \to \mathrm{H}^{n+1}(U, F|_U) \to \dots$$

which is functorial in F.

Proof. The sequence $0 \to j_! j^{-1} F \xrightarrow{\varepsilon} F \xrightarrow{\eta} i_* i^{-1} F \to 0$ is an exact sequence of sheaves on X where ε is the counit morphism of the adjunction $(j_!, j^{-1})$ and η is the unit morphism of the adjunction (i^{-1}, i_*). Taking the long exact cohomology sequence (Theorem 9.3 (3)) and identifying $\mathrm{H}^n(X, i_* i^{-1} F) = \mathrm{H}^n(Z, F|_Z)$ and $\mathrm{H}^n(X, j_! j^{-1} F) = \mathrm{H}^n(U, F)$ one gets the desired sequence. □

Proposition 9.15 (Local vanishing). *For every sheaf F on X and for every $n \geq 1$ the sheafification of the presheaf $X \supseteq U \mapsto \mathrm{H}^n(U, F)$ is zero.*

Proof. Let $F \hookrightarrow I^\bullet$ be an injective resolution so that

$$\mathrm{H}^n(U, F) = \frac{\ker\big(I^n(U) \to I^{n+1}(U)\big)}{\mathrm{im}\big(I^{n-1}(U) \to I^n(U)\big)}$$

for every open subset U of X. Since filtered colimits are exact, we get for every $x \in X$ on stalks that

$$\mathrm{H}^n(-, F)_x = \operatorname*{colim}_{x \in U} \mathrm{H}^n(U, F) = \operatorname*{colim}_{x \in U} \frac{\ker\big(I^n(U) \to I^{n+1}(U)\big)}{\mathrm{im}\big(I^{n-1}(U) \to I^n(U)\big)} = \frac{\ker\big(I^n_x \to I^{n+1}_x\big)}{\mathrm{im}\big(I^{n-1}_x \to I^n_x\big)}$$

which vanishes for $n \geq 1$ since the complex $I^\bullet$ is exact in degrees ≥ 1. □

Definition 9.16 (Godement sheaf). Let F be a sheaf on X. We define the presheaf $\mathcal{C}(F)$

$$\begin{aligned} \mathcal{C}(F)\colon \mathrm{Open}(X)^{\mathrm{op}} &\longrightarrow \mathrm{Mod}(R), \\ U &\mapsto \mathcal{C}(F)(U) := \bigoplus_{x \in U} F_x, \end{aligned}$$

where the restriction maps $\mathcal{C}(F)(U) \to \mathcal{C}(F)(V)$ for $V \subseteq U$ are induced by the indentity maps on summands $x \in V$ and the zero maps for summands $x \in U \setminus V$.

Exercise 9.17. Find a topological space $\tilde{X}$ and a continuous map $f: \tilde{X} \to X$ such that the morphism $F \to \mathcal{C}(F)$ identifies with the unit morphism of the adjunction (f^{-1}, f_*).

Lemma 9.18. *For any sheaf F on X the presheaf $\mathcal{C}(F)$ is a flasque sheaf and there exists a canonical monomorphism $F \hookrightarrow \mathcal{C}(F)$ of sheaves on X.*

Proof. Exercise. □

Construction 9.19 (Godement resolution). Let F be a sheaf on X so that we have a monomorphism $\varphi: F \hookrightarrow \mathcal{C}(F)$ into the Godement sheaf. Setting $\mathcal{C}^{-1}(F) := F$, $\mathcal{C}^0(F) := \mathcal{C}(F)$, and $d^{-1} := \varphi$ we can inductively define for $n \in \mathbf{N}$ the sheaf

$$\mathcal{C}^{n+1}(F) := \mathcal{C}\Big(\mathrm{coker}\big(\mathcal{C}^{n-1}(F) \xrightarrow{d^{n-1}} \mathcal{C}(F)\big)\Big)$$

so that we get an induced morphism $d^n: \mathcal{C}^n(F) \to \mathcal{C}^{n+1}(F)$. Check that the induced sequence

$$0 \to F \xrightarrow{\varphi} \mathcal{C}^0(F) \xrightarrow{d^0} \mathcal{C}^1(F) \xrightarrow{d^1} \mathcal{C}^2(F) \xrightarrow{d^2} \dots$$

is exact. By Lemma 9.18 we have constructed a flasque resolution $F \xrightarrow{\varphi} \mathcal{C}^\bullet(F)$, the so-called **Godement resolution**.

10. Čech cohomology

In this section let X be a topological space and let R be a ring. All sheaves in this section are sheaves of R-modules.

For a sheaf F on X, its sheaf cohomology $\mathrm{H}^*(X,F)$ is defined via "resolving" the sheaf F in terms of better behaving sheaves (Definition 9.2). Another approach is to "resolve" the space X.

Construction 10.1 (Čech complex)**.** Let F be a sheaf on X and let $\mathcal{U} := (U_i)_{i\in I}$ be an open cover of X. For elements $i_0,\dots,i_n \in I$ we write $U_{i_0,\dots,i_n} := U_{i_0}\cap\dots\cap U_{i_n}$. For $n\in\mathbf{N}$ we define the R-module

$$\check{C}^n(\mathcal{U},F) := \prod_{(i_0,\dots,i_n)\in I^{n+1}} F(U_{i_0,\dots i_1})$$

and the morphism

$$\begin{aligned} d^n\colon \check{C}^n(\mathcal{U},F) &\longrightarrow \check{C}^{n+1}(\mathcal{U},F),\\ s=(s_{i_0,\dots,i_n})_{(i_0,\dots,i_n)\in I^{n+1}} &\mapsto (d^n(s)_{i_0,\dots,i_{n+1}})_{(i_0,\dots,i_{n+1})\in I^{n+2}}\\ \text{where}\quad d^n(s)_{i_0,\dots,i_{n+1}} &:= \sum_{k=0}^{n+1}(-1)^k s_{i_0,\dots,i_{k-1},i_{k+1},\dots,i_{n+1}}|_{U_{i_0,\dots,i_{n+1}}}\end{aligned}$$

Check that the resulting sequence

$$\check{C}^0(\mathcal{U},F)\xrightarrow{d^0}\check{C}^1(\mathcal{U},F)\xrightarrow{d^1}\check{C}^2(\mathcal{U},F)\xrightarrow{d^2}\dots$$

is a complex, the **Čech complex** $\check{C}^\bullet(\mathcal{U},F)$ of the sheaf F with respect to the cover $\mathcal{U}$.

The first differential is the morphism

$$\begin{aligned} d^0\colon \check{C}^0(\mathcal{U},F)=\prod_{i\in I}F(U_i) &\longrightarrow \prod_{i,j\in I}F(U_i\cap U_j)=\check{C}^1(\mathcal{U},F)\\ (s_i)_{i\in I} &\mapsto (s_j|_{U_i\cap U_j}-s_i|_{U_i\cap U_j})_{i,j\in I}\end{aligned}$$

so that $F(X)\cong\ker(d^0)$ due to the sheaf condition.

Definition 10.2. Let F be a sheaf on X and let $\mathcal{U}=(U_i)_{i\in I}$ be a cover of X. For $n\in\mathbf{N}$ the **n-th Čech cohomology** of F with respect to $\mathcal{U}$ is defined to be

$$\check{\mathrm{H}}^n(\mathcal{U},F) := \mathrm{H}^n(\check{C}^\bullet(\mathcal{U},F)).$$

Lemma 10.3. *For a sheaf F on X and any cover $\mathcal{U}$ of X there is a canonical isomporphism*

$$F(X)\xrightarrow{\cong}\check{\mathrm{H}}^0(\mathcal{U},F).$$

Proof. This follows immedeately from Construction 10.1. □

Construction 10.4 (Sheaf Čech complex)**.** Let F be a sheaf on X and let $\mathcal{U}=(U_i)_{i\in I}$ be an open cover of X. For an open subset V of X we write $\mathcal{U}|_V := (U_i\cap V)_{i\in I}$ for the induced cover of V. Check that for every $n\in\mathbf{N}$ the induced presheaf $V\mapsto\check{C}^n(\mathcal{U}|_V,F|_V)$ is actually a sheaf (by using that F itself is a sheaf) which we will denote by $\underline{\check{C}}^n(\mathcal{U},F)$. The induced complex $\underline{\check{C}}^\bullet(\mathcal{U},F)$ of sheaves is called the **sheaf Čech complex**. By design we have that

$$\check{\mathrm{H}}^n(\mathcal{U},F)=\mathrm{H}^n\big(\underline{\check{C}}^\bullet(\mathcal{U},F)(X)\big).$$

Lemma 10.5. *For every sheaf F on X and every cover $\mathcal{U}$ of X the sheaf Čech complex*

$$0\longrightarrow F\longrightarrow\underline{\check{C}}^0(\mathcal{U},F)\longrightarrow\underline{\check{C}}^1(\mathcal{U},F)\longrightarrow\underline{\check{C}}^2(\mathcal{U},F)\longrightarrow\dots$$

is exact.

Proof. For an open subset U of X with inclusion map $j\colon U \hookrightarrow X$ we write $\underline{R}_U := j_! j^{-1}\underline{R}$ where. For an inclusion $U \subset V$ of open subsets of X there is a morphism $\underline{R}_U \to \underline{R}_V$. Hence for every open subset V of X there is a complex of sheaves

$$(\dagger) \qquad 0 \longleftarrow \underline{R}_V \xleftarrow{\delta_0} \bigoplus_{i\in I} \underline{R}_{V\cap U_i} \xleftarrow{\delta_1} \bigoplus_{(i,j)\in I^2} \underline{R}_{V\cap U_{i,j}} \xleftarrow{\delta_2} \cdots$$

where the differentials are defined analogously as for the Čech complex.[15] Applying $\underline{\mathrm{Hom}}(-,F)$ we get a complex of sheaves

$$0 \longrightarrow \underline{\mathrm{Hom}}(\underline{R}_V,F) \longrightarrow \underline{\mathrm{Hom}}(\bigoplus_{i\in I}\underline{R}_{V\cap U_i},F) \longrightarrow \underline{\mathrm{Hom}}(\bigoplus_{(i,j)\in I^2}\underline{R}_{V\cap U_{i,j}},F) \longrightarrow \cdots$$

and and using the universal property of the coproduct we get

$$0 \longrightarrow \underline{\mathrm{Hom}}(\underline{R}_V,F) \longrightarrow \prod_{i\in I}\underline{\mathrm{Hom}}(\underline{R}_{V\cap U_i},F) \longrightarrow \prod_{(i,j)\in I^2}\underline{\mathrm{Hom}}(\underline{R}_{V\cap U_{i,j}},F) \longrightarrow \cdots$$

which identifies with the complex

$$(\clubsuit) \qquad 0 \longrightarrow F(V) \longrightarrow \prod_{i\in I} F(V\cap U_i) \longrightarrow \prod_{(i,j)\in I^2} F(V\cap U_{i,j}) \longrightarrow \cdots$$

Claim. If V is connected and $V \subseteq U_\alpha$ and for some $\alpha \in I$, then the complex $(\dagger)$ is nullhomotopic.

Check that the claim implies that the complex $(\dagger)$ is exact and that this inherits to the complex $(\clubsuit)$. Conclude that this implies that the stalks of the Čech complex are exact, hence the Čech complex itself.

It remains to show the claim. Let U_α for some $\alpha \in I$. We want to construct a homotopy $(s_n)_{n\in\mathbf{N}}$ with

$$s_n\colon \bigoplus_{(i_0,\dots,i_{n-1})\in I^n} \underline{R}_{V\cap U_{i_0,\dots,i_{n-1}}} \longrightarrow \bigoplus_{(i_0,\dots,i_n)\in I^{n+1}} \underline{R}_{V\cap U_{i_0,\dots,i_n}}.$$

going from degree $n-1$ to degree n. Denote by $1_V \in \underline{R}_V(V) \cong R$ the unit elemet (here one uses that U is connected) and write $1_W \in \underline{R}_W(W)$ for the section induced by restricting 1_V. Then s_0 is uniquely define by setting $s_0(1_V) := 1_{V\cap U_\alpha}$ and for $n \geq 1$ we set

$$s_n(1_{V\cap U_{i_0,\dots,i_n}}) := 1_{V\cap U_{\alpha,i_0,\dots,i_n}}.$$

Check that for every $n \in \mathbf{N}$ we have $\delta_n \circ s_{n+1} + s_n \circ \delta_{n-1} = \mathrm{id}$. Thus $(\dagger)$ is nullhomotopic (Definition 4.12 (v))[16] and the claim follows. □

Corollary 10.6 (Local vanishing, cf. Proposition 9.15). *Let F be a sheaf on X and let $\mathcal{U}$ be a cover of X. Then for every $n \geq 1$ the sheafification of the presheaf $X \supseteq V \mapsto \check{\mathrm{H}}^n(\mathcal{U}|_V, F|_V)$ is zero.*

11. Sheaf cohomology and Čech cohomology

In this section let X be a topological space and let R be a ring. All sheaves in this section are sheaves of R-modules.

Theorem 11.1. *Let F be a sheaf on X and let $\mathcal{U}$ be a cover of X. Then there exists a canonical morphism*

$$\gamma_{\mathcal{U},F}\colon \check{\mathrm{H}}^n(\mathcal{U},F) \longrightarrow \mathrm{H}^n(X,F)$$

from Čech cohomology with respect to $\mathcal{U}$ to sheaf cohomology which is functorial in F.

[15] For convenience, we use here a different notation for the differentials: the differential δ_n has target in degree n (whereas usually the differential d^n has source in degree n).

[16] Be aware of the different notation for the differentials.

Proof. Let $F \hookrightarrow I^\bullet$ be an injective resolution of F. Since the Čech complex is exact (Lemma 10.5), there exists a morphsim of complexes $\gamma^\bullet_{\mathcal{U},F}\colon \underline{\check{C}}^\bullet(\mathcal{U},F) \to I^\bullet$ extending the identity morphism on F (Proposition 4.15). Thus we get a morphism

$$\check{\mathrm{H}}^n(\mathcal{U},F) = \mathrm{H}^n(\underline{\check{C}}^\bullet(\mathcal{U},F)(X)) \xrightarrow{\mathrm{H}^n(\gamma^\bullet_{\mathcal{U},F})} \mathrm{H}^n(I^\bullet(X)) = \mathrm{H}^n(X,F).$$

It remains to show that this is functorial in the sheaf. Given a morphism $\varphi\colon F \to G$ of sheaves on X and injective resolutions $F \hookrightarrow I^\bullet$ and $G \hookrightarrow J^\bullet$ there exists a morphism $\varphi^\bullet\colon I^\bullet \to J^\bullet$ extending φ (Proposition 4.15). By design of the Čech complex, there exists a morphism $\underline{\check{C}}^\bullet(\mathcal{U},\varphi)\colon \underline{\check{C}}^\bullet(\mathcal{U},F) \to \underline{\check{C}}^\bullet(\mathcal{U},G)$ also extending φ. It remains to show that the resulting diagram

$$\begin{tikzcd}
\underline{\check{C}}^\bullet(\mathcal{U},F) \arrow[r, "\underline{\check{C}}^\bullet(\mathcal{U},\varphi)"] \arrow[d, "\gamma^\bullet_{\mathcal{U},F}"'] & \underline{\check{C}}^\bullet(\mathcal{U},G) \arrow[d, "\gamma^\bullet_{\mathcal{U},G}"] \\
I^\bullet \arrow[r, "\varphi^\bullet"] & J^\bullet
\end{tikzcd}$$

induces a commuative diagram on cohomology groups. This is an exercise. □

Definition 11.2. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X. A **refinement** of $\mathcal{U}$ is an open cover $\mathcal{V} = (V_j)_{j\in J}$ such that for every $j \in J$ there exists an $i \in I$ such that $V_j \subset U_i$.

Lemma 11.3. *(i) For any two open covers $\mathcal{U}$ and $\mathcal{V}$ there exists a common refinement $\mathcal{W}$, i.e. $\mathcal{W}$ is both a refinement of $\mathcal{U}$ and a refinement of $\mathcal{V}$.*

(ii) Let F be a sheaf on X and let $n \in \mathbf{N}$. For an open cover $\mathcal{U}$ of X and every refinement $\mathcal{V}$ of $\mathcal{U}$ there exists an canonical morphism

$$\check{\mathrm{H}}^n(\mathcal{U},F) \longrightarrow \check{\mathrm{H}}^n(\mathcal{V},F).$$

Proof. (i) is an exercise. (ii) Let $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$. By the axiom of choice there exists a map of sets $\varphi\colon J \to I$ such that $V_j \subseteq U_{\varphi(j)}$ for every $j \in J$. Check that this yields a morphism of complexes $\varrho^\bullet_\varphi\colon \check{C}^\bullet(\mathcal{U},F) \to \check{C}^\bullet(\mathcal{V},F)$. Check that for another choice of a map $\varphi'\colon J \to I$ there exists a homotopy $\varrho^\bullet_\varphi \simeq \varrho_{\varphi'}$ so that the induced map on cohomology groups does not depend on the choice. □

Definition 11.4. Let F be a sheaf on X and $n \in \mathbf{N}$. We define the ***n*-th absolute Čech cohomology** of F to be

$$\check{\mathrm{H}}^n(X,F) := \operatorname*{colim}_{\mathcal{U}} \check{\mathrm{H}}^n(\mathcal{U},F)$$

where the colimit runs over all open covers $\mathcal{U}$ of X and the morphisms within the colimit are those of Lemma 11.3 so that the colimit is a filtered colimit.

The morphisms of morphism $\gamma_{\mathcal{U},F}$ of Theorem 11.1 induce a morphism

$$\gamma_F\colon \check{\mathrm{H}}^n(X,F) \longrightarrow \mathrm{H}^n(X,F)$$

by passing to the colimit over all open covers.

Definition 11.5. An open cover $(U_i)_{i\in I}$ of a topological space X is said to be **locally finite** iff every point $x \in X$ has an open neighbourhood V_x such that only finitely many of the U_i intersect V_x. The space X is called **paracompact** iff every open cover has a refinement which is locally finite.

Examples 11.6. (i) Every quasi-compact space is paracompact.

(ii) Every metric space is paracompact; this is a theorem.[17]

(iii) Stefan Schröer has constructed an example of a space which is hausdorff, but not paracompact [Sch13, §1].

[17] This is originally due to A. H. Stone, a newer and short proof is due to M. E. Rudin (link).

Theorem 11.7. *Assume that X is hausdorff and paracompact. Then the morphism*

$$\gamma_F : \check{\mathrm{H}}^n(X,F) \longrightarrow \mathrm{H}^n(X,F)$$

from Definition 11.4 is an isomorphism.

Proof strategy. We will show that both $(\check{\mathrm{H}}^n(X,-))_{n\in\mathbf{N}}$ and $(\mathrm{H}^n(X,-))_{n\in\mathbf{N}}$ are universal δ-functors (Definition 11.8) and hence they agree by a general statement (Theorem 11.10). □

Definition 11.8. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A **δ-functor** from $\mathcal{A}$ to $\mathcal{B}$ is for every $n \in \mathbf{N}$

- an additive functor $F^n : \mathcal{A} \to \mathcal{B}$ and
- for every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ a morphism
$$\delta^n : F^n(C) \longrightarrow F^{n+1}(A)$$
in $\mathcal{B}$

such that the following properties hold:

(i) For every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ the induced sequence
$$\begin{tikzcd} 0 \arrow[r] & F^0(A) \arrow[r, "F^0(f)"] & F^0(B) \arrow[r, "F^0(g)"] & F^0(C) \arrow[dll, "\delta^0"] \\ & F^1(A) \arrow[r, "F^1(f)"] & F^1(B) \arrow[r, "F^1(g)"] & F^1(C) \arrow[dll, "\delta^1"] \\ & F^2(A) \arrow[r, "F^2(f)"] & F^2(B) \arrow[r, "F^2(g)"] & \cdots\cdots \end{tikzcd}$$
in $\mathcal{B}$ is exact.

(ii) For every commutative diagram
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$
in $\mathcal{A}$ with exact rows and every $n \in \mathbf{N}$ we get a commuative diagram
$$\begin{tikzcd} F^n(A) \arrow[r] \arrow[d] & F^n(B) \arrow[r] \arrow[d] & F^n(C) \arrow[r, "\delta^n"] \arrow[d] & F^{n+1}(A) \arrow[d] \\ F^n(A') \arrow[r] & F^n(B') \arrow[r] & F^n(C') \arrow[r, "\delta^n"] & F^{n+1}(A'). \end{tikzcd}$$

A **morphism of δ-functors** $(F^n)_{n\in\mathbf{N}} \to (G^n)_{n\in\mathbf{N}}$ is a family of natural transformations $(\varphi_n : F^n \to G^n)_{n\in\mathbf{N}}$ such that for every $n \in \mathbf{N}$ and for every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ the induced diagram

$$\begin{tikzcd} F^n(C) \arrow[r, "\delta_F^n"] \arrow[d, "\varphi_n(C)"'] & F^{n+1}(A) \arrow[d, "\varphi_{n+1}(A)"] \\ G^n(C) \arrow[r, "\delta_G^n"'] & G^{n+1}(A) \end{tikzcd}$$

in $\mathcal{B}$ commutes.

A δ-functor $(F^n)_{n\in\mathbf{N}}$ is called a **universal δ-functor** iff for every δ-functor $(G^n)_{n\in\mathbf{N}}$ and for every natural transformation $\varphi: F^0 \to G^0$ there exists a unique morphism of δ-functors $(\varphi_n: F^n \to G^n)_{n\in\mathbf{N}}$ with $\varphi = \varphi_0$.

Definition 11.9. A functor $F: \mathcal{A} \to \mathcal{B}$ is called **effaceable** iff for every object A in $\mathcal{A}$ there exists a monomorphism $A \hookrightarrow I$ into an object I of $\mathcal{A}$ such that $F(I) = 0$.

Theorem 11.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories.*

(1) If $(F^n)_{n\in\mathbf{N}}$ and $(G^n)_{n\in\mathbf{N}}$ are universal δ-functors such that there exists a natural isomorphism $F^0 \cong G^0$, then there exists a unique natural isomorphism $(F^n)_{n\in\mathbf{N}} \cong (G^n)_{n\in\mathbf{N}}$ of δ-functors.
(2) A δ-functor $(F^n)_{n\in\mathbf{N}}$ from $\mathcal{A}$ to $\mathcal{B}$ is a universal δ-functor if and only if for every $n \geq 1$ the functor $F^n: \mathcal{A} \to \mathcal{B}$ is effaceable.
(3) For every left-exact functor $F: \mathcal{A} \to \mathcal{B}$ its right-derived functors $(\mathrm{R}^n F)_{n\in\mathbf{N}}$ form a universal δ-functor.

Proof. (1) is a standard argument. (2) is an exercise. (3) follows immediately from (2) and Theorem 4.9. $\square$

Corollary 11.11. *Sheaf cohomology $(\mathrm{H}^n(X,-))_{n\in\mathbf{N}}$ is a universal δ-functor.*

Now we want to show that Čech cohomology is a universal δ-functor if X is hausdorff and paracompact. We begin with some statements which hold for arbitrary topological spaces.

Lemma 11.12. *Let F be a sheaf on X let $\mathcal{U}$ be cover of X. If F is flasque, then $\check{\mathrm{H}}^n(\mathcal{U},F) = 0$ for $n \geq 1$. In particular, the functor $\check{\mathrm{H}}^n(\mathcal{U},-)$ is effaceable for every $n \geq 1$ (Definition 11.9).*

Proof. Let F be flasque and let $n \geq 1$. Then the sheaf $\underline{\check{C}}^n(\mathcal{U},F)$ is flasque as well. Hence the Čech complex is a flasque resolution of F so that $\check{\mathrm{H}}^n(\mathcal{U},F) = \mathrm{H}^n(\underline{\check{C}}^\bullet(\mathcal{U},F)) = \mathrm{H}^n(X,F) = 0$ since flasque sheaves are $\Gamma(X,-)$-acyclic (Theorem 9.10). $\square$

Now let us turn to long exact sequences associated with short exact sequences of sheaves.

Construction 11.13. Let $0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0$ be an exact sequence of sheaves on X. Since the section functor $\Gamma(V,-)$ for every open subset V of X is left-exact, we get an exact sequence of complexes

$$0 \longrightarrow \check{C}^\bullet(\mathcal{U},F) \xrightarrow{\check{C}^\bullet(\mathcal{U},f)} \check{C}^\bullet(\mathcal{U},G) \xrightarrow{\check{C}^\bullet(\mathcal{U},g)} \check{C}^\bullet(\mathcal{U},H)$$

Setting $\check{C}^\bullet_g(\mathcal{U},H) := \mathrm{im}\big(\check{C}^\bullet(\mathcal{U},g)\big)$, which is a subcomplex of $\check{C}^\bullet(\mathcal{U},H)$, we get an exact sequence of complexes

$$0 \longrightarrow \check{C}^\bullet(\mathcal{U},F) \xrightarrow{\check{C}^\bullet(\mathcal{U},f)} \check{C}^\bullet(\mathcal{U},G) \xrightarrow{\check{C}^\bullet(\mathcal{U},g)} \check{C}^\bullet_g(\mathcal{U},H) \longrightarrow 0$$

Setting $\check{\mathrm{H}}^n_g(\mathcal{U},H) := \mathrm{H}^n(\check{C}^\bullet_g(\mathcal{U},H))$ we get a long exact sequence

$$\ldots \longrightarrow \check{\mathrm{H}}^n(\mathcal{U},F) \xrightarrow{\check{\mathrm{H}}^n(\mathcal{U},f)} \check{\mathrm{H}}^n(\mathcal{U},G) \xrightarrow{\check{\mathrm{H}}^n(\mathcal{U},g)} \check{\mathrm{H}}^n_g(\mathcal{U},H) \xrightarrow{\partial} \check{\mathrm{H}}^{n+1}(\mathcal{U},F) \longrightarrow \ldots$$

by Proposition 4.17. Furthermore, for every $n \in \mathbf{N}$ there exists a morphism

$$\check{\mathrm{H}}^n_g(\mathcal{U},H) \longrightarrow \check{\mathrm{H}}^n(\mathcal{U},H)$$

which is induced by the inclusion of subcomplexes.

Proposition 11.14. *The morphism $\check{\mathrm{H}}^0_g(X,H) \to \check{\mathrm{H}}^0(X,H)$ is an isomorphism and the morphism $\check{\mathrm{H}}^1_g(X,H) \to \check{\mathrm{H}}^1(X,H)$ is a monomorphism. In particular, the sequence*

$$0 \to \check{\mathrm{H}}^0(X,F) \to \check{\mathrm{H}}^0(X,G) \to \check{\mathrm{H}}^0(X,H) \xrightarrow{\partial} \check{\mathrm{H}}^1(X,F) \to \check{\mathrm{H}}^1(X,G) \to \check{\mathrm{H}}^1(X,H)$$

is exact.

Proof. Exercise, use Lemma 11.15 below. □

Lemma 11.15. *Let $\mathcal{U} = (U_i)_{i\in I}$ and let $s = (s_i)_i \in \check{C}^0(\mathcal{U},H)$. Then there exists an open cover $\mathcal{V} = (V_j)_{j\in J}$ of X and a map $\varphi\colon J \to I$ such that $V_j \subseteq U_{\varphi(j)}$ for every $j \in J$ and such that $\varphi(s) := (s_{\varphi(j)}|_{V_j})_j \in \check{C}_g(\mathcal{V},H)$.*

Proof. This follows from unfolding what it means that the morphism of sheaves $g\colon G \to H$ is surjective. □

For the higher degrees we need that the space is hausdorff and paracompact.

Lemma 11.16 (cf. Lemma 11.15)**.** *Assume that X is hausdorff and paracompact. Let $n \in \mathbf{N}$, let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X, and let $s = (s_{i_0,\dots,i_n})_{(i_0,\dots,i_n)} \in \check{C}^n(\mathcal{U},H)$. Then there exists an open cover $\mathcal{V} = (V_j)_{j\in J}$ of X and a map $\varphi\colon J \to I$ such that $V_j \subseteq U_{\varphi(j)}$ for every $j \in J$ and such that $\varphi(s) := (s_{\varphi(j_0),\dots,\varphi(j_n)}|_{V_{j_0,\dots,j_n}})_{(j_0,\dots,j_n)} \in \check{C}^n_g(\mathcal{V},H)$.*

Proof. Exercise. □

Proposition 11.17. *If X is hausdorff and paracompact, then for every $n \in \mathbf{N}$ the morphism $\check{\mathrm{H}}^n_g(X,H) \to \check{\mathrm{H}}^n(X,H)$, which is the colimit of the morphism in Construction 11.13, is an isomorphism. In particular, the sequence*

$$\begin{tikzcd}
0 \arrow[r] & \check{\mathrm{H}}^0(X,F) \arrow[r] & \check{\mathrm{H}}^0(X,G) \arrow[r] & \check{\mathrm{H}}^0(X,H) \arrow[dll, "\partial"'] \\
& \check{\mathrm{H}}^1(X,F) \arrow[r] & \check{\mathrm{H}}^1(X,G) \arrow[r] & \check{\mathrm{H}}^1(X,H) \arrow[dll, "\partial"'] \\
& \check{\mathrm{H}}^2(X,F) \arrow[r] & \check{\mathrm{H}}^2(X,G) \arrow[r] & \cdots\cdots
\end{tikzcd}$$

is exact.

Proof. Exercise, use Lemma 11.16. □

Proof of Theorem 11.7 (sketch). According to Theorem 11.10 it suffices to show that the family $(\check{\mathrm{H}}^n(X,-))_{n\in\mathbf{N}}$ is a universal δ-functor. Lemma 11.12 implies that the functors $\check{\mathrm{H}}^n(X,-)$ are effacable for $n \geq 1$. The existence of long exact sequences is Propositon 11.17. It remains to show that these long exact sequences are functorial in diagrams of short exact sequence. This is omitted. □

Proposition 11.18. *Let F be a sheaf on X and let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X such that $\mathrm{H}^n(U_{i_0,\dots,i_k},F) = 0$ for every $(i_0,\dots,i_n) \in I^{k+1}$ for every $k \in \mathbf{N}$ and every $n \geq 1$. Then*

$$\check{\mathrm{H}}^n(\mathcal{U},F) \cong \mathrm{H}^n(X,F).$$

Proof. This is an exercise. *Hint: Use induction and proceed similarly as in the proof of Theorem 9.10.* □

12. First cohomology and computations

In this section let X be a topological space and let R be a ring. All sheaves in this section are sheaves of R-modules.

Lemma 12.1. *Let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X and let $\mathcal{V} = (V_j)_{j\in J}$ be a refinement of $\mathcal{U}$, i.e. there exists a map $\varphi\colon J \to I$ such that $V_j \subseteq U_{\varphi(j)}$ for every $j \in J$.*

(i) The morphism $\check{\mathrm{H}}^1(\mathcal{U},F) \longrightarrow \check{\mathrm{H}}^1(\mathcal{V},F)$ defined by

$$(s_{i_0,i_1})_{(i_0,i_1)\in I^2} \mapsto (s_{\varphi(j_0),\varphi(j_1)}|_{V_{j_0,j_1}})_{(j_0,j_1)\in J^2}$$

is independent of the choice of the map φ.

(ii) The morphism $\check{\mathrm{H}}^1(\mathcal{U},F) \longrightarrow \check{\mathrm{H}}^1(\mathcal{V},F)$ is injective.

Proof. (i) is a special case of Lemma 11.3. (ii) For convenience, we omit the index sets and the restrictions in the notation. Let $(s_{i_0,i_1})_{i_0,i_1}$ be in $\ker(\check{C}^1(\mathcal{U},F) \overset{d^1}{\to} \check{C}^2(\mathcal{U},F))$ and such that its image $(s_{\varphi(j_0),\varphi(j_1)})_{j_0,j_1}$ lies in $\mathrm{im}(\check{C}^0(\mathcal{V},F) \overset{d^0}{\to} \check{C}^1(\mathcal{V},F)) \subset \ker(\check{C}^1(\mathcal{V},F) \overset{d^1}{\to} \check{C}^2(\mathcal{V},F))$. We have to show that $(s_{i_0,i_1})_{i_0,i_1}$ lies in $\mathrm{im}(\check{C}^0(\mathcal{U},F) \overset{d^0}{\to} \check{C}^1(\mathcal{U},F))$.

By assumption there exists an element $(t_j)_j \in \check{C}^0(\mathcal{V},F) = \prod_{j\in J} F(V_j)$ such that $s_{\varphi(j_0),\varphi(j_1)} = t_{j_0} - t_{j_1}$ on V_{j_0,j_1} for every $j_0, j_1 \in J$.

Fixing $i \in I$ and restricting further to $U_i \cap V_{j_0,j_1}$ we get

$$t_{j_0} - t_{j_1} = s_{\varphi(j_0),\varphi(j_1)} = s_{\varphi(j_0),i} + s_{i,\varphi(j_0)} = s_{i,\varphi(j_1)} - s_{i,\varphi(j_0)},$$

where the second and the third equality follow as $(s_{i_0,i_1})_{i_0,i_1} \in \ker(d^1)$. Hence we have $s_{i,\varphi(j_0)} + t_{j_0} = s_{i,\varphi(j_1)} + t_{j_1}$ on $U_i \cap V_{j_0,j_1}$. Since $(U_i \cap V_j)_j$ is a cover of U_i we can glue the $(s_{i,j})_j$ to a section $\bar{s}_i \in F(U_i)$ so that $\bar{s}_i|_{V_j} = s_{i,\varphi(j)} + t_j$ in $F(U_i \cap V_j)$.

For every $i_0, i_1 \in I$ and every $j \in J$ we have after restricting to $U_{i_0,i_1} \cap V_j$ that

$$s_{i_0,i_1} = s_{i_0,\varphi(j)} + s_{\varphi(j),i_1} = (s_{i_0,\varphi(j)} + t_j) - (f_{i_1,\varphi(j)} + t_k) = \bar{s}_{i_0} - \bar{s}_{i_1}$$

and hence we have $s_{i_0,i_1} = \bar{s}_{i_0} - \bar{s}_{i_1}$ already on U_{i_0,i_1}. Thus $(s_{i_0,i_1})_{i_0,i_1} = d^0((\bar{s}_i)_i)$ which was to show. □

Corollary 12.2. *For every sheaf F on X and every cover $\mathcal{U}$ of X the canonical morphism*

$$\check{\mathrm{H}}^1(\mathcal{U},F) \longrightarrow \check{\mathrm{H}}^1(X,F)$$

is injective.

Theorem 12.3 (Leray). *Let F be a sheaf on X and let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X such that $\check{\mathrm{H}}^1(U_i,F) = 0$ for every $i \in I$. Then the canonical morphism*

$$\check{\mathrm{H}}^1(\mathcal{U},F) \longrightarrow \check{\mathrm{H}}^1(X,F)$$

is an isomorphism.[18]

Proof. Let $\mathcal{V} = (V_j)_{j\in J}$ be a refinement of $\mathcal{U}$ via a map $\varphi\colon J \to I$. It suffices to show that the morphism

$$\check{\mathrm{H}}^1(\mathcal{U},F) \longrightarrow \check{\mathrm{H}}^1(\mathcal{V},F)$$

is an isomorphism. By Lemma 12.1 it is injective.

[18]Compare to Proposition 11.18 where vanishing of higher sheaf cohomology was assumed.

In order to prove surjectivity, let $(t_{j_0,j_1})_{j_0,j_1}$ be an element of $\ker(\check{C}^1(\mathcal{V},F) \xrightarrow{d^1} \check{C}^2(\mathcal{V},F))$. We want to find a preimage $(s_{i_0,i_1})_{i_0,i_1}$ in $\ker(\check{C}^1(\mathcal{U},F) \xrightarrow{d^1} \check{C}^2(\mathcal{U},F))$, up to elements in $\mathrm{im}(\check{C}^0(\mathcal{V},F) \xrightarrow{d^0} \check{C}^1(\mathcal{V},F))$.

By assumption and Corollary 12.2, we have that $\check{\mathrm{H}}^1(\mathcal{V}|_{U_i},F) = 0$ for every $i \in I$. Hence there exists for every $i \in I$ an element $(r_{i,j})_j$ in $\prod_j F(U_i \cap V_j)$ such that

$$t_{j_0,j_1} = r_{i,j_0} - r_{i,j_1} \tag{$\spadesuit$}$$

after restricting to $U_i \cap V_{j_0,j_1}$. Restricting further to $U_{i_0,i_1} \cap V_{j_0,j_1}$ we get

$$r_{i_1,j_0} - r_{i_0,j_0} = r_{i_1,j_1} - r_{i_0,j_0}.$$

For every $i_0,i_1 \in I$, since $(U_{i_0,i_1} \cap V_j)_j$ is a cover of U_{i_0,i_1} we get a glued section s_{i_0,i_1} in $F(U_{i_0,i_1})$ so that

$$s_{i_0,i_1} = r_{i_1,j} - r_{i_0,j} \tag{$\clubsuit$}$$

on $U_{i_0,i_1} \cap V_j$ for every $j \in J$. Check that the element $(s_{i_0,i_1})_{i_0,i_1}$ lies in $\ker(d^1)$.

Using ($\clubsuit$) and ($\spadesuit$) we get on V_{j_0,j_1} that

$$s_{\varphi(j_0),\varphi(j_1)} - t_{j_0,j_1} = \big(r_{\varphi(j_1),j_0} - r_{\varphi(j_0),j_0}\big) - \big(r_{\varphi(j_1),j_0} - r_{\varphi(j_1),j_1}\big) = r_{\varphi(j_1),j_1} - r_{\varphi(j_0),j_0}$$

since the term $r_{\varphi(j_1),j_0}$ cancels out. Hence $(s_{\varphi(j_0),\varphi(j_1)} - t_{j_0,j_1})_{j_0,j_1} = d^0((r_{\varphi(j),j})_j)$ which was to show. □

In order to compute explicit examples, it is convenient to repalce the Čech complex with a "smaller" complex which has the same cohomology. This will be done by getting rid of redundant factors of the product $\check{C}^n(\mathcal{U},F) = \prod_{(i_0,\dots,i_n)\in I^{n+1}} F(U_{i_0,\dots,i_n})$.

Construction 12.4. Let F be a sheaf on X and let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X. We fix a total order "$<$" on the index set I.[19] For $n \in \mathbf{N}$ we define the R-module

$$\check{C}^n_{\mathrm{ord}}(\mathcal{U},F) := \prod_{i_0<\dots<i_n} F(U_{i_0,\dots,i_n})$$

which is a submodule of $\check{C}^n(\mathcal{U},F)$ from Construction 10.1. We define differentials

$$d^n : \check{C}^n_{\mathrm{ord}}(\mathcal{U},F) \longrightarrow \check{C}^{n+1}_{\mathrm{ord}}(\mathcal{U},F)$$

just as the restriction of the differentials on $\check{C}^\bullet(\mathcal{U},F)$; check that this is well-defined, i.e. that $d(s) \in \check{C}_{\mathrm{ord}}(\mathcal{U},F)$ if $s \in \check{C}_{\mathrm{ord}}(\mathcal{U},F)$. We call the complex $(\check{C}^\bullet_{\mathrm{ord}}(\mathcal{U},F),d^\bullet)$ the **ordered Čech complex** of the sheaf F with respect to the cover $\mathcal{U}$.

Now we want to construct a comparison morphism

$$c^\bullet : \check{C}^\bullet_{\mathrm{ord}}(\mathcal{U},F) \longrightarrow \check{C}^\bullet(\mathcal{U},F)$$

which is *different from the inclusion morphism*. In degree $n \in \mathbf{N}$ we define for $s = (s_{i_0,\dots,i_n})_{i_0<\dots<i_n}$ its image $c^n(s)$ via

$$c^n(s)_{i_0<\dots<i_n} := \begin{cases} 0 & \big(\exists k \neq l : i_k = i_l\big) \\ \mathrm{sgn}(\sigma)\cdot s_{i_{\sigma(0)},\dots,i_{\sigma(n)}} & \big(\exists \sigma \in \mathrm{Aut}(\{0,\dots,n\}) : i_{\sigma(0)} < \dots < i_{\sigma(n)}\big). \end{cases}$$

where $\mathrm{sgn}(\sigma)$ is the sign of the permutation; note that these two case are disjoint and exhaustive and that the permutation σ is uniquely defined (if it exists).

[19] This is always possible due to the well-ordering theorem.

Proposition 12.5. *The comparison morphism $c^\bullet\colon \check{C}^\bullet_{\mathrm{ord}}(\mathcal{U},F) \to \check{C}^\bullet(\mathcal{U},F)$ is a homotopy equivalence. In particular, for every $n \in \mathbf{N}$ the induced morphism*

$$\mathrm{H}^n(\check{C}^\bullet_{\mathrm{ord}}(\mathcal{U},F)) \xrightarrow{\mathrm{H}^n(c^\bullet)} \mathrm{H}^n(\check{C}^\bullet(\mathcal{U},F)) \overset{\mathrm{def}}{=} \check{\mathrm{H}}^n(\mathcal{U},F)$$

is an isomorphism.

Proof. There is morphism $\pi^\bullet\colon \check{C}^\bullet(\mathcal{U},F) \to \check{C}^\bullet_{\mathrm{ord}}(\mathcal{U},F)$ which forgets all the components which are not in the right order. One immediately gets that the composition $\pi^\bullet \circ c^\bullet$ equals the identity in $\check{C}^\bullet_{\mathrm{ord}}(\mathcal{U},F)$. One can show that the other composition $c^\bullet \circ \pi^\bullet$ is homotopic (Definition 4.12 (vi)) to the identity on $\check{C}^\bullet(\mathcal{U},F)$. This is outsourced, see [Stacks, Tag 01FM]. □

Now we can compute our first explicit example:

Example 12.6. For a sheaf F on $\mathbf{C}$ we want to compute the group $\mathrm{H}^1(\mathbf{C}^\times,F)$ from Proposition 1.2. Since $\mathbf{C}$ is a metric space, hence paracompact, we have that $\mathrm{H}^1(\mathbf{C}^\times,F) \cong \check{\mathrm{H}}^1(\mathbf{C}^\times,F)^1$ by Theorem 11.7.[20]. We set $X := \mathbf{C}^\times$ and consider the cover $X = U_1 \cup U_2$ with $U_1 := \mathbf{C}^\times \smallsetminus \mathbf{R}_{<0}$ and $U_2 := \mathbf{C}^\times \smallsetminus \mathbf{R}_{>0}$ so that $U_{1,2} = U_1 \cap U_2 = \mathbf{C} \smallsetminus \mathbf{R}$.

Claim. The spaces U_1 and U_2 are contractible.

By symmetry, it suffices to show that U_1 is contractible. The inclusion map $i\colon \{1\} \hookrightarrow U_1$ is a right-inverse to the unique map $p\colon U_1 \to \{1\}$, i.e. $p \circ i = \mathrm{id}_{\{1\}}$. We construct a homotopy between $i \circ p = \mathrm{const}_1$ and id_{U_1}. Check that the map

$$h\colon U_1 \times [0,1] \longrightarrow U_1, \quad x + \mathbf{i}y \mapsto (1 + t(x-1)) + \mathbf{i}ty$$

is well-defined and satifsies $h(z,0) = i \circ p(z)$ and $h(z,1) = \mathrm{id}_{U_1}(z)$ for all $z = x + \mathbf{i}y \in U$ $(x,y \in \mathbf{R})$. Hence U_1 is contractible.

Hence $\check{\mathrm{H}}^1(U_i,F) = 0$ for $i \in \{1,2\}$ by homotopy invariance (will be done later) and we get that $\check{\mathrm{H}}^1(X,F) \cong \check{\mathrm{H}}^1(\mathcal{U},F)$ and the latter is isomorphic to $\mathrm{H}^1(\check{C}^\bullet_{\mathrm{ord}}(\mathcal{U},F))$ by Proposition 12.5. Hence we have to compute the cohomology of the complex

$$\prod_{i \in \{1,2\}} F(U_i) \longrightarrow \prod_{i<j \in \{1,2\}} F(U_{i,j}) \longrightarrow \prod_{i<j<k \in \{1,2\}} F(U_{i,j,k})$$

which is

$$\begin{aligned} F(U_1) \times F(U_2) &\xrightarrow{d^0} F(U_{1,2}) \xrightarrow{d^1} 0 \\ (s_1,s_2) &\mapsto s_2|_{U_{1,2}} - s_1|_{U_{1,2}}. \end{aligned}$$

Everthing together yields $\mathrm{H}^1(\mathbf{C}^\times,F) = F(\mathbf{C} \smallsetminus \mathbf{R})/\mathrm{im}(d^0)$.

For $F = \underline{\mathbf{Z}}$ we have that $\underline{\mathbf{Z}}(\mathbf{C} \smallsetminus \mathbf{R}) \cong \mathbf{Z}^{\pi_0(\mathbf{C} \smallsetminus \mathbf{R})} \cong \mathbf{Z} \times \mathbf{Z}$ since $\mathbf{C} \smallsetminus \mathbf{R}$ has precisely two connected components (the upper half-plane and the lower half-plane). Similarly $F(U_1) \cong \mathbf{Z} \cong F(U_2)$. Hence

$$\mathrm{H}^1(\mathbf{C}^\times, \underline{Z}) = \mathbf{Z} \times \mathbf{Z} / \langle (y-x, y-x) | x,y \in \mathbf{Z} \rangle \cong \mathbf{Z}.$$

13. Torsors and first cohomology

In this section let X be a topological space.

Reminder 13.1. Let G be a group and let S be a set.

[20] In fact, the morphism $\check{\mathrm{H}}^1(X,F) \to \mathrm{H}^1(X,F)$ is an isomorphism for all topological spaces, also for non-paracompact ones, see Proposition 13.6 below.

- A G**-action** on S is a map of sets $G\times S\to S, (g,s)\mapsto g\cdot s$, such that the induced map $G\to \mathrm{Hom}_{\mathrm{Set}}(S,S), g\mapsto (s\mapsto g\cdot s)$, is a group homomorphism.
- The action is called **free** iff $g\cdot s\neq s$ for every $g\in G\smallsetminus\{e_G\}$ where e_G is the neutral element of G.[21]
- The action is called **transitive** iff for every $s,t\in S$ there exists a $g\in G$ such that $s=g\cdot t$. If the action is also free, then the element g is unique.
- A G**-torsor** is a G-action which is both free and transitive.
- If $G\times S\to S$ is a G-torsor, then for every choice $s\in S$ the induced map $G\to S, g\mapsto g\cdot s$, is a bijection.

Definition 13.2. Let G be a sheaf of groups on X. An G**-torsor** is a sheaf of sets F together with a morphism of sheaves of sets $G\times F\to F$, called **action**, such that

(i) the induced action $G(U)\times F(U)\to F(U)$ is a $G(U)$-torsor if $F(U)\neq\varnothing$, and
(ii) for every $x\in X$ the stalk F_x is not empty.

A **morphism of** G**-torsors** is a morphism of the underlying sheaves of sets which is compatible with the actions. A G-torsor is said to be **trivial** iff there exists an isomorphism of G-torsors to G itself equipped with the left-action coming from the group structure.

Lemma 13.3. *Let G be a sheaf of groups on X. A G-torsor F is trivial if and only if $F(X)\neq\varnothing$.*

Proof. Exercise. $\square$

Proposition 13.4. *Let G be a sheaf of abelian groups. Then there exists a bijection*

$$\mathrm{H}^1(X,G)\xleftrightarrow{\simeq}\{G\text{-torsors on } X\}/\cong.$$

Proof. Let $\alpha\in\mathrm{H}^1(X,G)$. We want to construct a G-torsor F_α. Let $G\hookrightarrow I$ be a monomorphism into an injective sheaf so that we get an exact sequence

$$0\longrightarrow G(X)\longrightarrow I(X)\xrightarrow{\pi}(I/G)(X)\longrightarrow \mathrm{H}^1(X,G)\longrightarrow \underbrace{\mathrm{H}^1(X,I)}_{=0}$$

and hence a preimage $\beta\in(I/G)(X)$ of α. We define a subsheaf $F_\alpha\subset I$ by

$$F_\alpha(U):=\{s\in I(U)\mid \pi(s)=\beta|_U\}.$$

We can define an action by

$$G(U)\times F_\alpha(U)\longrightarrow F_\alpha(U),\quad (g,s)\mapsto g+s.$$

Check that this action is free and transitive so that F_α is a G-torsor.

Now let F be an G-torsor. There is an induced abelian sheaf $\underline{\mathbf{Z}}[F]$ which is defined as the sheafification of the presheaf $\underline{\mathbf{Z}}[F]^{\mathrm{pre}}$ which assigns an open subset U of X the free abelian group

$$\underline{\mathbf{Z}}[F]^{\mathrm{pre}}(U):=\bigoplus_{s\in F(U)}\mathbf{Z}\cdot s.$$

Then the morphism $\sigma\colon\underline{\mathbf{Z}}[F]\to\underline{\mathbf{Z}}, \sum_{i=1}^n k_i\cdot s_i\mapsto\sum_{i=1}^n k_i$. Its kernel $\ker(\sigma)$ is generated by sections of the form $s-t$ for sections s and t of F. There is a morphism of sheaves

$$\varphi\colon\ker(\sigma)\longrightarrow G,\quad s-t\mapsto g_{s,t}\ \text{such that}\ g_{s,t}+s=t.$$

[21]If the group G is abelian, we write the action additively $(g,s)\mapsto g+s$.

Hence we can form the pushout $E := G \oplus_{\ker(\sigma)} \underline{\mathbf{Z}}[F]$ so that we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(\sigma) \arrow[r] \arrow[d, "\varphi"] & \underline{\mathbf{Z}}[F] \arrow[r, "\sigma"] \arrow[d] & \underline{\mathbf{Z}} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & G \arrow[r] & E \arrow[r] & \underline{\mathbf{Z}} \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Then the image of $1_X \in \underline{\mathbf{Z}}(X) = \mathrm{H}^0(X, \underline{\mathbf{Z}})$ under the boundary morphism of the long exact cohomology sequence yields an element $\alpha_G \in \mathrm{H}^1(X, G)$.

It remains to check that these assignments are inverse to each other, i.e. that $\alpha = \alpha_{F_\alpha}$ for every $\alpha \in \mathrm{H}^1(X,G)$ and that $F \cong F_{\alpha_F}$ as G-torsors for every G-torsor F. □

Lemma 13.5. *Let G be sheaf of groups on X and let F be a G-torsor. Then F is locally-trivial, i.e. there exist an open cover $(U_i)_{i \in I}$ of X such that for evry $i \in I$ the $G|_{U_i}$-torsor $F|_{U_i}$ is trivial.*

Proof. Exercise. □

Proposition 13.6. *For every abelian sheaf G on X and every cover $\mathcal{U} = (U_i)_{i \in I}$ of X the morphism*

$$\check{\mathrm{H}}^1(\mathcal{U}, G) \longrightarrow \mathrm{H}^1(X, G)$$

is injective. The image of the postcompositon

$$\check{\mathrm{H}}^1(\mathcal{U}, G) \longrightarrow \mathrm{H}^1(X, G) \longrightarrow \{G\text{-torsors on } X\}/\cong$$

with the bijection of Proposition 13.4 is precisely the set of isomorphism classes of G-torsors F such that $F|_{U_i}$ is trivial for every $i \in I$.

Proof. We construct an inverse map from the set

$$\{\, F \in \mathrm{Sh}(X, \mathbf{Z}) \mid F \text{ is a } G\text{-torsor on } X \,\&\, \forall i \in I : F|_{U_i} \text{ is trivial} \,\}$$

to $\check{\mathrm{H}}^1(\mathcal{U}, G)$. Let F be such a G-torsor. Then, by Lemma 13.3, there exist a section $s_i \in F(U_i)$ for every $i \in I$. For every $i, j \in I$ there exists a unique section $g_{i,j} \in G(U_i \cap U_j)$ such that

$$g_{i,j} + s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

in $F(U_i \cap U_j)$ since the action is free and transitive. Check that $(g_{i,j})_{(i,j) \in I^2}$ lies in the kernel of the first differential $d^1 : \check{C}^1(\mathcal{U}, G) \to \check{C}^2(\mathcal{U}, G)$ of the Čech complex so that we get an element

$$\alpha_F \in \check{\mathrm{H}}^1(\mathcal{U}, G) = \frac{\ker\left(\check{C}^1(\mathcal{U}, G) \xrightarrow{d^1} \check{C}^2(\mathcal{U}, G)\right)}{\mathrm{im}\left(\check{C}^0(\mathcal{U}, G) \xrightarrow{d^0} \check{C}^1(\mathcal{U}, G)\right)}.$$

It remains to check that this assignment is indeed an inverse to the map in question. □

Corollary 13.7. *For every abelian sheaf G on X there exist an isomorphism*

$$\check{\mathrm{H}}^1(X, G) \longrightarrow \mathrm{H}^1(X, G).$$

Proof. This follows directly form Proposition 13.6 and Lemma 13.5. □

14. Higher direct images

In this section let R be a ring. All sheaves are sheaves of R-modules.

Definition 14.1. Let $f\colon X \to Y$ be a continuous map. Then the functor

$$f_*\colon \mathrm{Sh}(X,R) \longrightarrow \mathrm{Sh}(Y,R)$$

is a right-adjoint, hence is preserves limits, so it is left-exact. For every $n \in \mathbf{N}$ we define the **n-th higher direct image**

$$\mathrm{R}^n f_*\colon \mathrm{Sh}(X,R) \longrightarrow \mathrm{Sh}(Y,R).$$

Example 14.2. For the canonical map $\pi_X\colon X \to \{\star\}$ to the one-point space, we have for every sheaf F on X that

$$\mathrm{R}^n(\pi_X)_*F = \mathrm{H}^n(X,F)$$

since $(\pi_X)_* = \Gamma(X,-)$ as functors $\mathrm{Sh}(X,R) \to \mathrm{Sh}(\{\star\},R) \cong \mathrm{Mod}(R)$.

Lemma 14.3. *Let $f\colon X \to Y$ be a continuoius map and let F be a sheaf on X. For every $n \in \mathbf{N}$, the sheaf $\mathrm{R}^n f_*F$ is the sheafification of the presheaf*

$$Y \supseteq V \mapsto \mathrm{H}^n(f^{-1}(V),F).$$

Proof. Let $F \hookrightarrow I^\bullet$ be an injective resolution. Then $\mathrm{R}^n f_*F$ is by definition the cohomology sheaf $\underline{\mathrm{H}}^n(f_*I^\bullet)$. The latter is by definition the sheafification of the presheaf

$$Y \supseteq V \mapsto \frac{\ker\big(f_*I^n(V) \to f_*I^{n+1}(V)\big)}{\mathrm{im}\big(f_*I^{n-1}(V) \to f_*I^n(V)\big)} = \mathrm{H}^n(I^\bullet(f^{-1}(V)) = \mathrm{H}^n(f^{-1}(V),F).$$

□

Corollary 14.4. *Let $f\colon X \to Y$ be a continuous map. Then flasque sheaves on X are f_*-acyclic. In particular, one can compute the functors $(\mathrm{R}^n f_*)_{n\in\mathbf{N}}$ by using flasque resolutions.*

In order to deal with higher direct images of compositions of continuous maps, we will later use the following properties:

Theorem 14.5. *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be left-exact functors bewteen abelian categories and assume that both $\mathcal{A}$ and $\mathcal{B}$ have enough injective objects. Then we have for every $n \in \mathbf{N}$:*

(i) If G is exact, then $\mathrm{R}^n(G \circ F) = G \circ \mathrm{R}^n F$.
(ii) If F is exact, then there exists a natural transformation

$$\mathrm{R}^n(G \circ F) \longrightarrow (\mathrm{R}^n G) \circ F.$$

(iii) Assume that F sends injective objects in $\mathcal{A}$ to G-acyclic objects in $\mathcal{B}$. Then for every F-acyclic object A in $\mathcal{A}$ there exists a canonical isomorphism

$$\mathrm{R}^n(G \circ F)(A) \cong \mathrm{R}^n G(F(A)).$$

Proof. Exercise. □

15. Homotopy invariance

Definition 15.1. Let X and Y be topological spaces.

(i) Two maps $f_0, f_1 : X \to Y$ are called **homotopic** iff there exists a map $h : X \times [0,1] \to Y$ such that $f_0 = h \circ \iota_0$ and $f_1 = h \circ \iota_1$ where $\iota_k : X \to X \times [0,1], x \mapsto (x,k)$ is the inclusion map:

$$\begin{tikzcd}
X \arrow[dr, "\iota_0"'] \arrow[drr, bend left, "f_0"] & & \\
& X \times [0,1] \arrow[r, dotted, "\exists h"] & Y \\
X \arrow[ur, "\iota_1"] \arrow[urr, bend right, "f_1"'] & &
\end{tikzcd}$$

In this case, we write $f_0 \simeq f_1$. The map h is called a **homotopy** between f_0 and f_1.

(ii) A map $f : X \to Y$ between is called a **homotopy equivalence** iff there exists a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In this case, we write $X \simeq Y$.

(iii) The space X is called **contractible** iff X is homotopy equivalent to the one-point space, i.e. $X \simeq *$.

Lemma 15.2. *Let X and Y be topological spaces.*

(i) The relation of being homotopic induces an equivalence relation on the set $\mathrm{Map}(X,Y)$ of (continuous) maps between X and Y.

(ii) If X is contractible, then any two maps $f, g : X \to Y$ are homotopic.

(iii) If Y is contractible, then any two maps $f, g : X \to Y$ are homotopic.

Examples 15.3. Let $n \in \mathbf{N}$.

(i) $\mathbf{R}^n \simeq *$

(ii) The inclusion $\mathbf{S}^n \hookrightarrow \mathbf{R}^{n+1} \setminus \{0\}$ is a homotopy equivalence.

Proof. Exercise. $\square$

Definition 15.4. The **homotopy category of topological spaces** HoTop is the category with objects $\mathrm{obj}(\mathrm{HoTop}) := \mathrm{obj}(\mathrm{Top})$ and with morphism $[X,Y] := \mathrm{Map}(X,Y)/\simeq$ are given by homotopy classes of maps between X and Y for every pair of topological spaces X and Y. Check that this yiels indeed a well-defined category.

Reminder 15.5. A sheaf F on X is said to be **constant** iff $F \cong \underline{M}$ for some R-module M. A sheaf is called **locally constant** iff there exist an open cover $(U_i)_{i \in I}$ of X such that for every $i \in I$ the sheaf $F|_{U_i}$ is a constant sheaf on U_i.

Lemma 15.6. *Let $f : X \to Y$ be a continuous map. Then for every locally constant sheaf G on Y, the inverse image sheaf $f^{-1}G$ is locally constant.*

Proof. Exercise. $\square$

Theorem 15.7. *Let $f_0, f_1 : X \to Y$ be homotopic maps and let G be a locally constant sheaf on Y. Then for every $n \in \mathbf{N}$ there exists a morphism*

$$\alpha : \mathrm{H}^n(X, f_0^{-1}G) \longrightarrow \mathrm{H}^n(X, f_1^{-1}G)$$

such that the diagram

$$\begin{tikzcd}
& \mathrm{H}^n(Y,G) \arrow[dl, "f_0^*"'] \arrow[dr, "f_1^*"] & \\
\mathrm{H}^n(X, f_0^{-1}G) \arrow[rr, "\alpha"] & & \mathrm{H}^n(X, f_1^{-1}G)
\end{tikzcd}$$

commutes where the morphisms f_0^* and f_1^* are those of Proposition 9.13 (i).

Proof. Later. □

Lemma 15.8. *Let K be a compact and relatively hausdorff* [22] *subspace of X. Then for every $n \in \mathbf{N}$ the canonical morphism*

$$\operatorname*{colim}_{K \subset U \text{ open}} \mathrm{H}^n(U,F) \longrightarrow \mathrm{H}^n(K,F|_K)$$

is an isomorphism.

Proof. We first consider the case where $n = 0$. Since the global sections functor is left-exact, the morphism

$$(\heartsuit) \qquad \operatorname*{colim}_{K \subset U \text{ open}} F(U)) \longrightarrow F|_K(K)$$

is injective. In order to show surjectivity, let $s \in F|_K(K)$. Since the morphism of sheaves $\operatorname{colim}_{K\subset U} F|_U \to F|_K$ is an isomorphism, there exists a family $(U_i)_{i\in I}$ of open subsets of X such that $K \subset \bigcup_{i\in I} U_i$ and there exists sections $s_i \in F|_U(U) = F(U)$ such that $s_i|_{K\cap U_i} = s|_{K\cap U_i}$. Since K is compact, we may assume that I is finite. There exists another family $(V_i)_{i\in I}$ such that $K \subset \bigcup_{i\in I} V_i$ and such that $K \cap \overline{V}_i \subset U_i$ for every $i \in I$.

Claim. There exists an open neighbourhood of K on which the sections $(s_i)_{i\in I}$ glue together.

By induction we can assume that $I = \{1,2\}$. Setting $K_i = K \cap \overline{V}_i$ we have $s_1|_{K_1\cap K_2} = s_2|_{K_1\cap K_2}$ by design. Hence we find an open neighbourhood W of $K_1 \cap K_2$ such that $s_1|_W = s_2|_W$. Since K is relatively hausdorff, there exists open subsets W_1 and W_2 such that

$$K_i \smallsetminus W \subset W_i \text{ for } i \in \{1,2\} \quad \text{and} \quad W_1 \cap W_2 = \varnothing.$$

Setting $U_i' := W_i \cup W$ we get that $s_1|_{U_1'\cap U_2'} = s_2|_{U_1'\cap U_2'}$ so that we can glue and get a section $s' \in F(U_1' \cup U_2')$ such that $s = s'|_K$, hence the morphism in question is surjective so that we get the morphism $(\heartsuit)$ is an isomorphism.

For the general case let $F \hookrightarrow I^\bullet$ be an injective resolution. Using the case where $n = 0$ we get for every $n \geq 0$ that

$$\mathrm{H}^n(K,F|_K) = \mathrm{H}^n(I^\bullet|_K(K)) = \mathrm{H}^n(\operatorname*{colim}_{K\subset U} I^\bullet(U)) = \operatorname*{colim}_{K\subset U} \mathrm{H}^n(I^\bullet(U)) = \operatorname*{colim}_{K\subset U} \mathrm{H}^n(U,F)$$

since cohomology of complexes commutes with filtered colimits. □

Lemma 15.9. *Let F be a sheaf on $[0,1]$. Then:*

(i) For $n \geq 2$ we have $\mathrm{H}^n([0,1],F) = 0$.
(ii) If the morphism $F([0,1]) \to F_t$ is an epimorphism for every $t \in [0,1]$, then $\mathrm{H}^1([0,1],F) = 0$.

Proof. Let $n \geq 0$. For every $0 \leq t_1 \leq t_2 \leq 1$ we have restriction isomorphisms

$$f_{t_1,t_2}\colon \mathrm{H}^n([0,1],F) \to \mathrm{H}^n([t_1,t_2],F|_{[t_1,t_2]}).$$

For $\alpha \in \mathrm{H}^n([0,1],F)$ we consider the set

$$J_\alpha := \{t \in [0,1] \mid f_{0,t}(\alpha) = 0\}.$$

- We have $0 \in J_\alpha$ since $\mathrm{H}^n(\{0\},F|_{\{0\}}) = 0$.
- If $t \in J_\alpha$, then $[0,t] \subset J_\alpha$ by design.
- Thus J_α is an intervall.
- J_α is open, since $\mathrm{H}^n([0,t],F|_{[0,t]}) = \operatorname{colim}_{t<s} \mathrm{H}^n([0,s],F|_{[0,s]})$ by Lemma 15.8.

[22] That is that for all $x,y \in K$ with $x \neq y$ there exists disjoint open neighbourhood U_x and U_y of x resp. y in X.

Claim. J_α is closed. (This implies that $J_\alpha = [0,1]$ since $[0,1]$ is connected.)

For $0 \le t \le s$ we have the Mayer-Vietoris sequence (Proposition 9.12)

$$\ldots \to \mathrm{H}^n([0,s],F|_{[0,s]}) \to \mathrm{H}^n([0,t],F|_{[0,t]}) \oplus \mathrm{H}^n([t,s],F|_{[t,s]} \to \mathrm{H}^n(\{t\},F_{\{t\}}) \to \ldots$$

Hence we get that

$$(\clubsuit) \qquad \mathrm{H}^n([0,s],F|_{[0,s]}) \cong \mathrm{H}^n([0,t],F|_{[0,t]}) \oplus \mathrm{H}^n([t,s],F|_{[t,s]})$$

if

(i) $n \ge 2$ or if
(ii) the morphism $\mathrm{H}^0([0,t],F|_{[0,t]}) = F([0,1]) \to F_t = \mathrm{H}^0(\{t\},F|_{\{t\}})$ is surjective.

Setting $s := \sup(J_\alpha)$ we get that

$$\operatorname*{colim}_{t<s} \mathrm{H}^n([t,s],F|_{[t,s]}) = \mathrm{H}^n(\{s\},F|_{\{s\}}) = 0.$$

Taking the colimit of the isomorphism ($\clubsuit$) we get

$$\mathrm{H}^n([0,s],F|_{[0,s]}) = \operatorname*{colim}_{t<s} \mathrm{H}^n([0,s],F|_{[0,s]}) \cong \operatorname*{colim}_{t<s} \mathrm{H}^n([0,t],F|_{[0,t]}).$$

Hence $f_{0,s}(\alpha) = 0$ since $f_{0,t}(\alpha) = 0$ for all $t < s$. Thus $s \in J_\alpha$ so that J_α is closed. □

Lemma 15.10. *Let $p: X \times [0,1] \to X$ be the projection map. Then for every locally constant sheaf F on $X \times [0,1]$ the counit morphism $p^{-1}p_*F \to F$ is an isomorphism.*

Proof. The assertion can be checked on stalks, so let $(x,t) \in X \times [0,1]$. Then we have isomorphisms

$$(p^{-1}p_*F)_{(x,t)} \overset{(1)}{=} (p_*F)_x \overset{(2)}{=} \Gamma(\{x\},(p_*F)|_{\{x\}}) \overset{(3)}{=} \Gamma(\underbrace{\{x\} \times [0,1]}_{=p^{-1}(\{x\})}, F|_{\{x\}\times[0,1]})$$
$$\overset{(4)}{=} F_{(x,t)}$$

where (1) is true in general, (2) holds since both sides have the same definition , (3) holds by defintion of the pushforward, and (4) follows as a locally constant sheaf on $[0,1]$ is already constant. □

Lemma 15.11. *Let $f: X \to Y$ be a continuous map and let $n \in \mathbf{N}$. Then postcomposing the natural transformation*

$$\mathrm{R}^n((\pi_X)_* \circ f^{-1}) \longrightarrow (\mathrm{R}^n(\pi_X)_*) \circ f^{-1}$$

from Theorem 14.5 (ii) with the natural transformation

$$\mathrm{R}^n(\pi_Y)_* \longrightarrow \mathrm{R}^n((\pi_Y)_* \circ f_* \circ f^{-1}) = \mathrm{R}^n((\pi_X)_* \circ f^{-1})$$

and then plugging in a sheaf G on Y identifies with the morphism

$$f^*: \mathrm{H}^n(Y,G) \to \mathrm{H}^n(X,f^{-1}G)$$

from Proposition 9.13 (i).

Proof. Exercise. □

Lemma 15.12. *Let $f: X \to Y$ and $g: Y \to Z$ be continuous maps. Then for every sheaf H on Z we have that*

$$f^* \circ g^* = (g \circ f)^* : \mathrm{H}^n(Z,H) \to \mathrm{H}^n(X,(g \circ f)^{-1}H).$$

Proof. Exercise. □

Lemma 15.13. *Let $p: X \times [0,1] \to X$ be the projection map and let G be a sheaf on $X \times [0,1]$. Then for every $x \in X$ we have that*

$$(\mathrm{R}^n p_* G)_x \cong \mathrm{H}^n(\{x\} \times [0,1], G|_{\{x\}\times[0,1]}).$$

Proof. Let $G \hookrightarrow J^\bullet$ be an injective resolution of sheaves on $X \times [0,1]$. Then we have

$$\begin{aligned}(\mathrm{R}^n p_* G)_x &= \underline{\mathrm{H}}^n(p_* J^\bullet))_x = \mathrm{H}^n((p_* J^\bullet)_x) \\ &\overset{(1)}{=} \mathrm{H}^n(\operatorname*{colim}_{x\in U} J^\bullet(U \times [0,1])) = \operatorname*{colim}_{x\in U} \mathrm{H}^n(U \times [0,1], G) \\ &\overset{(2)}{=} \mathrm{H}^n(\{x\} \times [0,1], G|_{\{x\}\times[0,1]})\end{aligned}$$

where (1) follows from cofinality and (2) is Lemma 15.8. □

Lemma 15.14. *Let F be a sheaf on X. Then the unit map $F \to p_* p^{-1} F$ is an isomorphism and $(\mathrm{R}^n p_*) p^{-1} F = 0$ for $n \geq 1$. In particular, the sheaf $p^{-1}F$ is p_*-acyclic.*

Proof. By Lemma 15.12 we have for every $t \in [0,1]$ that $((\mathrm{R}^n p_*) p^{-1} F)_x$ is isomorphic to

$$\mathrm{H}^n(\{x\} \times [0,1], (p^{-1}F)|_{\{x\}\times[0,1]}) = \begin{cases} (p^{-1}F)_{(x,t)} = F_x & (n = 0), \\ 0 & (n \geq 1) \text{ (by Lemma 15.9).} \end{cases}$$

Applying Lemma 15.9 for $n = 1$ we need that for every $t \in [0,1]$ the morphism

$$((p^{-1}F)|_{\{x\}\times[0,1]})(\{x\} \times [0,1]) \to ((p^{-1}F)|_{\{x\}\times[0,1]})_{(x,t)}$$

is an epimorphism; this follows since $(p^{-1}F)|_{\{x\}\times[0,1]}$ just the pullback of F along the constant morphism $\{x\} \times [0,1] \to X, (x,t) \mapsto x$. □

Proposition 15.15. *Let $p: X \times [0,1] \to X$ be the projection map. Then for every locally constant sheaf F on X the canonical morphism*

$$p^* : \mathrm{H}^n(X, F) \longrightarrow \mathrm{H}^n(X \times [0,1], p^{-1}F)$$

is an isomorphism. Moreover, for every $t \in [0,1]$ the inclusion map $i_t: X \cong X \times \{t\} \hookrightarrow X \times [0,1]$ induces an isomorphism

$$i_t^* : \mathrm{H}^n(X \times [0,1], p^{-1}F) \longrightarrow \mathrm{H}^n(X, F)$$

which is independent of the choice of $t \in [0,1]$.

Proof. By Lemma 15.14 the sheaf $p^{-1}F$ is p_*-acyclic. By Theorem 14.5 (iii) and we have

$$\begin{aligned}\mathrm{H}^n(X \times [0,1], p^{-1}F) &= \mathrm{R}(\pi_{X\times[0,1]}(p^{-1}F) = \mathrm{R}^n((\pi_X)_* p_*)(p^{-1}F) \\ &\cong \mathrm{R}^n(\pi_X)_*(p_* p^{-1}F) = \mathrm{H}^n(X, F)\end{aligned}$$

since $p_* p^{-1} F \cong F$ by Lemma 15.10.

Since $p \circ i_t = \mathrm{id}_X$ for all $t \in [0,1]$ we have that $i_t^* \circ p^*$ is the identity on $\mathrm{H}^n(X, F)$. Since p^* is an isomorphism, i_t^* is its inverse and hence unique. □

Proof of Theorem 15.7. By assumption we have a commutative diagram

$$\begin{tikzcd}
X \arrow[dr, "\iota_0"'] \arrow[drr, bend left, "f_0"] & & \\
& X \times [0,1] \arrow[r, "h"] & Y \\
X \arrow[ur, "\iota_1"] \arrow[urr, bend right, "f_1"'] & &
\end{tikzcd}$$

which yields for every $n \in \mathbf{N}$ a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^n(X, f_0^{-1}G) & & \\
& \mathrm{H}^n(X \times [0,1], h^{-1}G) \arrow[ul, "\iota_0*"', "\cong"] \arrow[dl, "\iota_1^*", "\cong"'] & \mathrm{H}^n(Y,G) \arrow[l, "h^*"'] \arrow[ull, bend right, "f_0^*"'] \arrow[dll, bend left, "f_1^*"] \\
\mathrm{H}^n(X, f_1^{-1}G) & &
\end{tikzcd}$$

where the morphism ι_0^* and ι_1^* are isomorphisms by Proposition 15.15 since the sheaves $f_0^{-1}G$ and $f_1^{-1}G$ are locally constant. Thus $\alpha := \iota_1^* \circ (\iota_0^*)^{-1}$ does the job. □

16. Sheaves in terms of étalé spaces

In this section let X be a topological space. All sheaves are sheaves of sets.

We want to examine the relation between sheaves on X and topological spaces over X.

Definition 16.1. A **topological space over** X is a pair (E,p) consisting of a topological space E together with a continuous map $p: E \to X$; for a point $x \in X$ we write $E_x := p^{-1}(\{x\})$ for the **fibre** of x. A **morphism** $f:(E,p) \to (F,q)$ between topological spaces over X is a continuous map $f: E \to F$ such that $p = q \circ f$. We get the category $\mathrm{Top}_{/X}$ of topological spaces over X.

Let (E,p) be a topological space over X. We define a presheaf of sets $\Gamma(E,p)$ whose sections over an open subset U of X are given by

$$\Gamma(E,p)(U) := \{s: U \to E \mid \forall x \in U : p \circ s(x) = x\}.$$

An element of $\Gamma(E,p)(U)$ is called a **section of** p over U. An element of $\Gamma(E,p)(X)$ is called a **global section of** p.[23]

Lemma 16.2. *For every topological space (E,p) over X the presheaf $\Gamma(E,p)$ is a sheaf and there is a canonical functor*

$$\Gamma: \mathrm{Top}_{/X} \longrightarrow \mathrm{Sh}(X), \quad (E,p) \mapsto \Gamma(E,p)$$

Proof. Exercise. □

Definition 16.3. For a topological space (E,p) over X call $\Gamma(E,p)$ the **sheaf of sections** of (E,p).

For a space (E,p) over X and a point $x \in X$ we are interested in the relation between the fibre E_x and the stalk $\Gamma(E,p)_x$. The following example show possible pathologies that can occur:

Example 16.4. Let $X = \{s, \eta\}$ be the two-point space whose open sets are precisely $\varnothing$, $\{\eta\}$ and X. For every topological space E equipped with the map $p: E \to \{s\} \hookrightarrow X$ the sheaf $\Gamma(E,p)$ is the empty sheaf, i.e. all sets of sections are empty.

In order to rule out such pathologies, we make the following definition:

Definition 16.5. A map of topological spaces $p: E \to X$ is said to be a **local homeomorphism** iff for every point $e \in E$ there exists an open neighbourhood V of e such that $p(V)$ is open in X and the induced map $p|_V : V \to p(V)$ is a homeomorphism. In particular, p is an open map. Denote by $\mathrm{LH}(X)$ the full subcategory of $\mathrm{Top}_{/X}$ spanned by local homeomorphisms.

[23]In fact, this is where the term "section" comes from.

Lemma 16.6. *If $p: E \to X$ is a local homeomorphism, then for every $x \in X$ there is a bijection*

$$\varphi: \Gamma(E,p)_x \xrightarrow{\cong} E_x.$$

Proof. Every element of $\Gamma(E,p)_x$ is of the form s_x for a section $s \in \Gamma(E,p)(U)$, i.e. a section $s: U \to E$, for an open neighbourhood U of x. Define $\varphi(s_x) := s(x)$ which is well-defined. In order to show injectivity, let $s,t: U \to E$ of p such that $s(x) = t(x)$ for some open neighbourhood U of x. Since p is a local homeomorphism, there exists an open neighbourhood V of $s(x) = t(x)$ such that $p|_V: V \to p(V)$ is an isomorphism. We may assume that $V \subseteq U$ so that both $s|_{p(V)}$ and $t|_{p(V)}$ are inverses of the isomorphism $p|_V$, hence they agree. For surjectivity one proceeds similarly. □

Construction 16.7. Let F be a presheaf of sets on X. Consider the disjoint union

$$\acute{\mathrm{E}}(F) := \coprod_{x \in X} F_x.$$

For an open subset U of X and a section $s \in F(U)$ we have a map

$$\acute{\mathrm{E}}(-,s): U \longrightarrow \acute{\mathrm{E}}(F), \quad x \mapsto \acute{\mathrm{E}}(x,s) := s_x.$$

We equip the set $\acute{\mathrm{E}}(F)$ with the topology generated by the basis

$$\{\mathrm{im}(\acute{\mathrm{E}}(U,s)) \mid U \subseteq X \text{open } \& \ s \in F(U)\}$$

so that $\acute{\mathrm{E}}(F)$ is a topological space. By desigin, the maps $\acute{\mathrm{E}}(-,s)$ are continuous and open. The space $\acute{\mathrm{E}}(F)$ is a space over X via the continuous (!) map

$$p_F: \acute{\mathrm{E}}(F) \to X$$

which sends $s \in F_x \subset \acute{\mathrm{E}}(F)$ to $x \in X$. Check that we get an induced functor

$$\acute{\mathrm{E}}(-): \mathrm{PSh}(X) \longrightarrow \mathrm{Top}_{/X}, \quad F \mapsto \acute{\mathrm{E}}(F).$$

Definition 16.8. For a presheaf F on X the topological space $(\acute{\mathrm{E}}(F), p_F)$ over X is called the **étalé space** of F.

Lemma 16.9. *For every presheaf F on X the morphism $p_F: \acute{\mathrm{E}}(F) \to X$ from Construction 16.7 is a local homeomorphism.*

Proof. For $e \in \acute{\mathrm{E}}(F)$ there exists an $x \in X$ such that $e \in F_x$. Hence there exists an open neighbourhood U of x and a section $s \in F(U)$ such that $e = s_x$. Then the continuous map $\acute{\mathrm{E}}(-,s): U \to E$ is injective and open, hence a homeomorphism onto its image. □

Lemma 16.10. *For every presheaf F on X, there is a canonical morphism of presheaves*

$$\eta_F: F \longrightarrow \Gamma(\acute{\mathrm{E}}(F), p_F)$$

which identifies $\Gamma(\acute{\mathrm{E}}(F), p_F)$ as the sheafification of F. Moreover, for every $x \in X$ the stalk F_x is isomorphic to $\acute{\mathrm{E}}(F)_x := p_F^{-1}(\{x\})$.

Proof. Exercise. *Hint: either you can check the universal property of the sheafification or you construct an explicit isomorphism using the explicit description of the sheafification.* □

Lemma 16.11. *Let $p: E \to X$ be a space over X. Then there exists a continuous map*

$$\varepsilon_E: \acute{\mathrm{E}}(\Gamma(E,p)) \longrightarrow E$$

of spaces over X. Moreover, if the map p is a local homeomorphism, then the map ε_E is a homeomorphism.

Proof. In order to construct ε_E, it suffices to construct for every $x \in X$ a continuous map $\Gamma(E,p)_x \to E$. So let $s\colon U \to E$ be a section of p on an open neighbourhood U of x. Set $\varepsilon(s_x) := s(x)$. For an open subset V of E, let $e \in \varepsilon_E^{-1}(V)$, hence there exists an $x \in X$ and a split $s\colon U \to E$ of p for an open neighbourhood U of x such that $e = s_x$ and such that $s(x) \in V$. Since s is continuous, the preimage $s^{-1}(V)$ is open in U and in X. Thus $e \in \acute{\mathrm{E}}(s^{-1}(V), s) \subset \varepsilon_E^{-1}(V)$ so that the latter is open. Hence ε_E is continuous.

If p is a local homeomorphism, then the map ε_E is a bijection on every fibre of $x \in X$, hence it is a bijection. Check that ε_E is also an open map. Thus it is a homeomorphism. □

Theorem 16.12. *The functor of sheaves of sections Γ from Lemma 16.2 and the étalé space functor $\acute{\mathrm{E}}(-)$ from Construction 16.7 yield an adjunction*

$$\acute{\mathrm{E}}(-)\colon \mathrm{PSh}(X) \rightleftarrows \mathrm{Top}_{/X} \colon \Gamma$$

which restricts to inverse equivalences of categories

$$\acute{\mathrm{E}}(-)\colon \mathrm{Sh}(X) \overset{\cong}{\rightleftarrows} \mathrm{LH}(X) \colon \Gamma$$

Proof. The unit $\eta\colon \mathrm{id}_{\mathrm{PSh}(X)} \to \Gamma \circ \acute{\mathrm{E}}(-)$ is given by the morphism η_F from Lemma 16.10 and the counit $\varepsilon\colon \acute{\mathrm{E}}(-) \circ \Gamma \to \mathrm{id}_{\mathrm{LH}(X)}$ is given by the morphism ε_E from Lemma 16.11. These lemmas also ensure that the triangle identities hold so that we get the desired adjunction. Moreover, after restricting to the full subcategories $\mathrm{Sh}(X)$ and $\mathrm{LH}(X)$ we get an induced adjunction where both the unit and the counit are isomorphisms. Hence both $\acute{\mathrm{E}}(-)$ and Γ are fully faithful, thus equivalences of categories. □

17. Étalé spaces of locally constant sheaves

In this section let X be a topological space. All sheaves are sheaves of sets.

We have an equivalence of categories between sheaves on X and local homeomorphisms over X (Theorem 16.12). We want to investigate which subcategory of $\mathrm{LH}(X)$ corresponds to the category of locally constant sheaves.

Definition 17.1. Denote by $\mathrm{LCSh}(X)$ the full subcategory of $\mathrm{Sh}(X)$ spanned by locally constant sheaves and by $\mathrm{CSh}(X)$ the full subcategory of constant sheaves.

Lemma 17.2. *Let F be a locally constant sheaf over X with étalé space $p_F\colon \acute{\mathrm{E}}(F) \to X$. If U is an open subset of X such that $F|_U \cong \underline{S}$ for a set S, then there exists a homeomorphism*

$$\begin{tikzcd}
p_F^{-1}(U) \arrow[dr, "p_F"'] & & U \times S \arrow[ll, "\varphi", "\cong"'] \arrow[dl, "\mathrm{pr}_1"] \\
& U &
\end{tikzcd}$$

where the set S is considered as a topological spcae with the discrete topology and $U \times S$ carries the product topology; this implies that $U \times S \cong \coprod_S U$ is the disjoint union of $|S|$-many copies of U.

Proof. Let U be an open subset of X such that $F|_U \cong \underline{S}$ for some set S. We may assume that U is connected. Then $F(U) \cong S$ and for every $x \in X$, the map $F(U) \to F_x$ is an isomorphism. Hence every $s \in S$ determines a unique section $\acute{\mathrm{E}}(-,s)\colon U \to \acute{\mathrm{E}}(F), x \mapsto s_x$. Define the map

$$\varphi\colon U \times S \to p_F^{-1}(U), \quad (x,s) \mapsto \acute{\mathrm{E}}(x,s).$$

Check that this map does the job. □

Definition 17.3. A continuous map $p: E \to X$ is said to be a **fibre bundle** iff there exists a topological space F such that every point $x \in X$ has an open neighbourhood U such that there exists a homeomorphism

$$\begin{tikzcd} p^{-1}(U) \arrow[dr, "p"'] & & U \times F \arrow[ll, "\varphi", "\cong"'] \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

where $U \times F$ carries the product topology. Denote by $\mathrm{Bun}(X)$ the full subcategory of $\mathrm{Top}_{/X}$ which is spanned by fibre bundles.

A fibre bundle $p: E \to X$ is called a **covering** iff for every $x \in X$ the fibre $E_x = p^{-1}(\{x\})$ is a discrete space with respect to the subspace topology inherited from E. Denote by $\mathrm{Cov}(X)$ the full subcategory of $\mathrm{Top}_{/X}$ which is spanned by coverings.

Lemma 17.4. *Let $p: E \to X$ be fibre bundle.*

(i) Then the topological space F in Definition 17.3 is unique up to homeomorphism.
(ii) For every $x \in X$, there exists a homeomorphism $E_x \cong F$.
(iii) If p is a covering, then it is a local homeomorphism.

Proof. Exercise. □

Definition 17.5. Let $p: E \to X$ be a local homeomorphism. We call the topological space F from Definition 17.3 the **fibre** of p.

Remark 17.6. Lemma 17.4 implies that we have a inclusions of categories

$$\begin{tikzcd} & \mathrm{LH}(X) \arrow[dr, hook] & \\ \mathrm{Cov}(X) \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{Top}_{/X} \\ & \mathrm{Bun}(X). \arrow[ur, hook] & \end{tikzcd}$$

Exercise 17.7. Find a local homeomorphism that it not a fibre bundle and find a fibre bundle that is not a local homeomorphism.

Theorem 17.8. *The functor of sheaves of sections Γ from Lemma 16.2 and the étalé space functor $\acute{\mathrm{E}}(-)$ from Construction 16.7 restrict to inverse equivalences of categories*

$$\acute{\mathrm{E}}(-): \mathrm{LCSh}(X) \overset{\cong}{\rightleftarrows} \mathrm{Cov}(X) : \Gamma.$$

Exercise 17.9. Charaterise the full subcategory of $\mathrm{Cov}(X)$ which corresponds under the equivalences from Theorem 17.8 to the category of constant sheaves. Find a locally constant sheaf that is not constant.

18. The fundamental groupoid

In this section let X be a topological space.

Reminder 18.1. Let $x_0 \in X$. The **fundamental group** of X with base point x_0 is the set $\pi_1(X,x_0) := [(S^1,1),(X,x_0)]_*$ of pointed homotopy classes of pointed maps from the circle S^1 to X, i.e. loops in X with fixed base point x_0. The group structure is given by concatenation of loops. Every path $\gamma\colon[0,1]\to X$ from $x_0=\gamma(0)$ to $x_1=\gamma(1)$ induces an isomorphism

$$\gamma_*\colon \pi_1(X,x_0) \xrightarrow{\cong} \pi_1(X,x_1).$$

The following notion is a generalisation of the fundamental group which is indendent of the choice of a base point.

Definition 18.2. We define a category $\Pi_1(X)$ whose objects are the points of X and whose morphism between two points $x,y\in X$ are

$$\mathrm{Hom}_{\Pi_1(X)}(x,y) := \left\{\gamma\colon[0,1]\to X \;\middle|\; \gamma(0)=x \;\&\; \gamma(1)=y\right\}/\simeq,$$

i.e. homotopy classes of paths from x to y. Composition is given by concatenation of paths.[24] It is immediate that the category $\Pi_1(X)$ is a groupoid, i.e. every morphism is an isomorphism. We call $\Pi_1(X)$ the **fundamental groupoid** of X

Reminder 18.3. The topological space X is said to be...

(i) **path-connected** iff for every $x,y\in X$ there exists a path $\gamma\colon[0,1]\to X$ with $\gamma(0)=x$ and $\gamma(1)=y$.
(ii) **simply connected** iff $\pi_1(X,x_0)\cong\{1\}$ for every point $x_0\in X$.
(iii) **locally path-connected** iff for every point $x\in X$ and every neighbourhood U od x there exists a neighbourhood V of x which is both contained in U and path-connected.

Lemma 18.4. *If X is simply connected, then any two paths γ and δ with $\gamma(0)=\delta(0)$ and $\gamma(1)=\delta(1)$ are homotopic.*

Proof. Exercise. □

Lemma 18.5.
(i) For every $x\in X$ we have that $\pi_1(X,x)=\mathrm{Aut}_{\Pi_1(X)}(x)$.
(ii) If X is path-connected and simply connected, then the canonical functor

$$p\colon \Pi_1(X)\longrightarrow\{*\}$$

to the trivial category with one object is an equivalence.
(iii) For every continuous map $f\colon X\to Y$ there exists a canonical functor

$$f_*\colon \Pi_1(X)\longrightarrow\Pi_1(Y).$$

(iv) For two topological spaces X and Y the canonical morphism

$$\mathrm{pr}_{X,*}\times\mathrm{pr}_{Y,*}\colon \Pi_1(X\times Y)\longrightarrow\Pi_1(X)\times\Pi_1(Y)$$

is an isomorphism if both X and Y are locally path-connected.

Proof. Exercise. □

19. Complements on (locally) constant sheaves

In this section let X be a topological space. All sheaves are sheaves of sets.

Reminder 19.1. A sheaf F on X is said to be **constant** iff there exists an isomorphism $F\cong\underline{S}$ for some set S. A sheaf is called **locally constant** iff there exist an open cover $(U_i)_{i\in I}$ of X such that for every $i\in I$ the sheaf $F|_{U_i}$ is a constant sheaf on U_i.

[24]Here one has to check that everything is well-defined, of course.

Lemma 19.2. *The functor*

$$(\pi_X)^{-1}\colon \mathrm{Set} = \mathrm{Sh}(\{*\}) \longrightarrow \mathrm{Sh}(X)$$

is fully faithful with essential image $\mathrm{CSh}(X)$.

Lemma 19.3. *Let $f\colon X \to Y$ be a continuous map.*

(i) For every locally constant (resp. constant) sheaf G on Y the inverse image sheaf $f^{-1}G$ is locally constant (resp. constant).

(ii) For every constant sheaf F on X the direct image f_*F is constant.

Exercise 19.4. Find a continuous map $f\colon X \to Y$ and a locally constant sheaf F on X such that the direct image f_*F is not locally constant.

Lemma 19.5. *Let F be a sheaf on X. If there exists an open cover $X = U_1 \cap U_2$ such that*

- *$F|_{U_1}$ and $F|_{U_2}$ are constant sheaves and*
- *$U_1 \cap U_2$ is nonempty and connected,*

then F is a constant sheaf.

Proof. By assumption, there exist sets S_1 and S_2 and isomorphisms

$$\varphi_i\colon \underline{S_i} \xrightarrow{\cong} F|_{U_i}$$

for $i \in \{1,2\}$. Since $U_1 \cap U_2$ is nonempty and connected, have

$$S_1 \cong F|_{U_1}(U_1 \cap U_2) = F|_{U_2}(U_1 \cap U_2) \cong S_2$$

We fix this isomorphism and set $S := S_1 = S_2$ so that we get isomorphisms

$$\chi_i\colon \underline{S}|_{U_i} \xrightarrow{\cong} F|_{U_i}$$

for $i \in \{0,1\}$. By the sheaf condition, the morphisms χ_1 and χ_2 glue to a morphism of sheaves $\chi\colon \underline{S} \to F$ which is an isomorphism on every stalk, hence an isomorphism. □

Corollary 19.6. *Every locally constant sheaf on the intervall $[0,1]$ is constant.*[25]

Lemma 19.7 (cf. Lemma 15.10)**.** *Assume that every locally constant sheaf on X is constant. Then for every topological space Y and every locally constant sheaf F on the product $X \times Y$ the counit morphism*

$$(p_Y)^{-1}(p_Y)_*F \longrightarrow F$$

is an isomorphism where $p_Y\colon X \times Y \to Y$ is the projection map.

Proof. This works the same way as in the proof of Lemma 15.10. □

Lemma 19.8. *If X and Y are topological spaces on which every locally constant sheaf is constant, then every locally constant sheaf on $X \times Y$ is constant.*

Proof. Let F be a locally constant sheaf on $X \times Y$ and let $(U_i)_{i \in I}$ be a cover of $X \times Y$ such that $F|_{U_i}$ is a constant sheaf for every $i \in I$. Consider the projection maps

$$X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y.$$

By definition of the product topology we may assume that $I = J \times K$ and that

$$U_i = p_X^{-1}(V_j) \times p_Y^{-1}(W_k)$$

for every $i = (j,k)$. Now we fix $k \in K$ and set

$$F_k := F|_{p_Y^{-1}(W_k)}.$$

[25] This result was already used in the proof of Lemma 15.10.

By design, for every $j \in J$ the sheaf $F_{k,j} := F_k|_{p_X^{-1}(V_j)}$ is constant. Since direct images preserve constant sheaves (Lemma 19.3), the sheaves $(p_X)_* F_{k,j}$ are constant on V_j for every $j \in J$. Check that we have

$$((p_X)_* F_k)|_{V_j} = ((p_X)_* F_{k,j})|_{V_j}$$

for every $j \in J$. Hence $(p_X)_* F_k$ is a locally constant sheaf on X, thus it is constant by assumtion. By Lemma 19.7, the counit morphism

$$(p_X)^{-1}(p_X)_* F_k \longrightarrow F_k$$

is an isomorphism, hence F_k is a constant sheaf on $X \times W_k$ and $(p_Y)_* F_k$ is a constant sheaf on W_k.

Now changing the roles of X and Y we see that $(p_Y)_* F$ is a constant sheaf on Y. Using again Lemma 19.7 we see that the counit morphism

$$(p_Y)^{-1}(p_Y)_* F \longrightarrow F$$

is an isomorphism and hence F is a constant sheaf as desired. □

Corollary 19.9. *For every $n \in \mathbf{N}$, every locally constant sheaf on $[0,1]^n$ is constant.*

20. Mondodromy

In this section let X be a topological space.

Reminder 20.1. Let G be a group and $\mathcal{C}$ be a category. A **G-representation** in $\mathcal{C}$ is a pair (X, ϱ) of an object C of $\mathcal{C}$ and a group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(C)$.[26] A **morphism** of representations $f:(C,\varrho) \to (D,\sigma)$ is a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C,D)$ such that $f \circ \varrho = \sigma \circ f$. Denote by $\mathrm{Rep}(G,\mathcal{C})$ the category of G-representations in $\mathcal{C}$. A G-representation (C,ϱ) is called **trivial** iff $\varrho = \mathrm{id}_C$. Denote by $\mathrm{Rep}_0(G,\mathcal{C})$ the full subcategory of $\mathrm{Rep}(G,\mathcal{C})$ which is spanned by trivial G-representations.

Lemma 20.2. *Let G be a group and let $\mathcal{C}$ be a category.*

(i) The functor

$$\mathcal{C} \longrightarrow \mathrm{Rep}(G,\mathcal{C}), \quad C \mapsto (C, \mathrm{id}_C)$$

is fully faithful with essential image $\mathrm{Rep}_0(G,\mathcal{C})$.

(ii) Considering G as a category with one object whose endomorphisms are G, then

$$\mathrm{Rep}(G,\mathcal{C}) \cong \mathrm{Fun}(G,\mathcal{C}).$$

(iii) Every group homomorphism $f: G \to H$ induces a functor

$$f^*: \mathrm{Rep}(H,\mathcal{C}) \longrightarrow \mathrm{Rep}(G,\mathcal{C}), \quad (C,\varrho) \mapsto (C, \varrho \circ f).$$

If f^* is an isomorphism, then f is an isomorphism.[27]

(iv) If $\mathcal{C}$ is an abelian category, then $\mathrm{Rep}(G,\mathcal{C})$ is abelian as well.

Proof. Exercise. □

Definition 20.3. Let $\mathcal{G}$ be a groupoid and let $\mathcal{C}$ be a category. A **$\mathcal{G}$-representation** in $\mathcal{C}$ is a functor $\mathcal{G} \to \mathcal{C}$. Write

$$\mathrm{Rep}(\mathcal{G},\mathcal{C}) := \mathrm{Fun}(\mathcal{G},\mathcal{C})$$

for the category of $\mathcal{G}$-representations in $\mathcal{C}$. A representation $\varrho: \mathcal{G} \to \mathcal{C}$ is called **trivial** iff ϱ is isomorphic to a constant functor $\mathcal{G} \to \mathcal{C}, G \mapsto (C, \mathrm{id}_C)$ for some object C in $\mathcal{C}$. Denote by

$$\mathrm{Rep}_0(\mathcal{G},\mathcal{C})$$

[26] Syntactically, a representation is the same thing a group action but the semantics is different. A group action focuses on object the group acts on, whereas a representation focuses on the group itself.

[27] If $\mathcal{C}$ is a small category, this means that the functor $\mathrm{Grp}^{\mathrm{op}} \to \mathrm{Cat}, G \mapsto \mathrm{Rep}(G,\mathcal{C})$ is conservative.

the full subcategory of $\mathrm{Rep}(\mathcal{G},\mathcal{C})$ which is spanned by trivial representations.

Lemma 20.4. *Let $\mathcal{G}$ be a groupoid and let $\mathcal{C}$ be a category.*

(i) The functor
$$\mathcal{C} \longrightarrow \mathrm{Rep}(\mathcal{G},\mathcal{C}), \quad C \mapsto (C, \mathrm{id}_C)$$
is fully faithful with essential image $\mathrm{Rep}_0(\mathcal{G},\mathcal{C})$.

(ii) Every functor $f\colon \mathcal{G} \to \mathcal{H}$ into another groupoid $\mathcal{H}$ induces a functor
$$f^*\colon \mathrm{Rep}(\mathcal{H},\mathcal{C}) \longrightarrow \mathrm{Rep}(\mathcal{G},\mathcal{C}), \quad (C,\varrho) \mapsto (C, \varrho \circ f).$$
If f^* is an isomorphism, then f is an isomorphism.

(iii) If $\mathcal{C}$ is an abelian category, then $\mathrm{Rep}(\mathcal{G},\mathcal{C})$ is abelian as well.

Proof. Exercise. □

In the sequel we are interested in $\Pi_1(X)$-representations.

Lemma 20.5. *Assume that X is locally path-connected and let F be a locally constant sheaf on X.*

(i) Every path $\gamma\colon [0,1] \to X$ induces an isomorphism
$$\mu(F,\gamma)\colon F_{\gamma(0)} \xrightarrow{\cong} F_{\gamma(1)}.$$

(ii) If $\gamma \simeq \delta$ are two homotopic paths in X with $\gamma(0) = \delta(0)$ and $\gamma(1) = \delta(1)$, then $\mu(F,\gamma) = \mu(F,\delta)$.

(iii) If γ and δ are two paths in X such that $\gamma(1) = \delta(0)$, then
$$\mu(F,\gamma\cdot\delta) = \mu(F,\delta) \circ \mu(F,\gamma)$$
where $\gamma\cdot\delta$ is the concatenation of γ and δ.

(iv) The assignment $F \mapsto \mu(F,\mu)$ defines a functor
$$\mu\colon \mathrm{LCSh}(X) \longrightarrow \mathrm{Rep}(\Pi_1(X), \mathrm{Set})$$
$$F \mapsto \big([\gamma] \mapsto \mu(F,\gamma)\big)$$

Proof. (i) The inverse image $\gamma^{-1}F$ is a locally constant sheaf on $[0,1]$, hence it is constant. Thus define $\mu(F,\gamma)$ to be the composition
$$F_{\gamma(0)} = (\gamma^{-1}F)_0 \xleftarrow{\cong} \gamma^{-1}F([0,1]) \xrightarrow{\cong} (\gamma^{-1}F)_1 = F_{\gamma(1)}.$$

(ii) Let $h\colon [0,1]^2 \to X$ be a homotopy between γ and δ such that

- $h(t,0) = \gamma(t)$ for all $t \in [0,1]$,
- $h(t,1) = \delta(t)$ for all $t \in [0,1]$,
- $h(0,s) = \gamma(0) = \delta(0)$ for all $s \in [0,1]$, and
- $h(1,s) = \gamma(1) = \delta(1)$ for all $s \in [0,1]$.

By Corollary 19.9 the sheaf $h^{-1}F$ is a constant sheaf on $[0,1]^2$. Hence we get a commutative diagram

$$\begin{tikzcd}
F_{\gamma(0)} = (\gamma^{-1}F)_0 & \gamma^{-1}F([0,1]) \arrow[l, "\cong"'] \arrow[r, "\cong"] & (\gamma^{-1}F)_1 = F_{\gamma(1)} \\
(h^{-1}F)_{(0,0)} = (h^{-1}F)_{(0,1)} \arrow[u, "="] \arrow[d, "="'] & h^{-1}F([0,1]^2) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[u, "\cong"'] \arrow[d, "\cong"] & (h^{-1}F)_{(1,0)} = (h^{-1}F)_{(1,1)} \arrow[u, "="'] \arrow[d, "="] \\
F_{\delta(0)} = (\delta^{-1}F)_0 & \delta^{-1}F([0,1]) \arrow[l, "\cong"'] \arrow[r, "\cong"] & (\delta^{-1}F)_1 = F_{\delta(1)}.
\end{tikzcd}$$

where the outer vertical maps yield the identitiy maps.

(iii) Exercise.

(iv) Let $f\colon F \to G$ be a morphism of sheaves into a locally constant sheaf G. Check that for every path $\gamma\colon[0,1]\to X$ the diagram

$$\begin{tikzcd} F_{\gamma(0)} \arrow[r, "\mu(F,\gamma)"] \arrow[d, "f_{\gamma(0)}"'] & F_{\gamma(1)} \arrow[d, "f_{\gamma(1)}"] \\ G_{\gamma(0)} \arrow[r, "\mu(G,\gamma)"] & G_{\gamma(1)} \end{tikzcd}$$

commutes. □

Definition 20.6. The functor

$$\mu\colon \mathrm{LCSh}(X) \longrightarrow \mathrm{Rep}(\Pi_1(X),\mathrm{Set})$$
$$F \mapsto \big([\gamma] \mapsto \mu(F,\gamma)\big)$$

is called the **monodromy functor**.

Proposition 20.7. *If X is locally path-connected and nonempty, then the monodromy functor restricts to an equivalence of categories*

$$\mu_0\colon \mathrm{CSh}(X) \longrightarrow \mathrm{Rep}_0(\Pi_1(X),\mathrm{Set}).$$

Proof. Precomposing with the equivalence $(\pi_X)^{-1}$ from Lemma 19.2 we obtain the functor

$$\mathrm{Set} = \mathrm{Sh}(\{*\}) \xrightarrow{(\pi_X)^{-1}} \mathrm{CSh}(X) \xrightarrow{\mu_0} \mathrm{Rep}_0(\Pi_1(X),\mathrm{Set})$$

from Lemma 20.4 which is an equivalence. Hence μ_0 is an equivalence. □

Theorem 20.8. *If X is locally path-connected, then the monodromy functor*

$$\mu\colon \mathrm{LCSh}(X) \longrightarrow \mathrm{Rep}(\Pi_1(X),\mathrm{Set})$$

is fully faithful.

Proof. **Faithful:** Let $f,g \in \mathrm{Hom}_{\mathrm{LCSh}(X)}(F,G)$ for objects F and G in $\mathrm{LCSh}(X)$ such that $\mu(f) = \mu(g) \in \mathrm{Hom}_{\mathrm{Rep}}(\mu(F),\mu(G))$. Then for every $x \in X$, the constant path $\gamma_x\colon[0,1]\to X, t \mapsto x$ yields equal morphisms $\mu(f,\gamma) = \mu(g,\gamma)\colon F_x \to G_x$, hence $f = g$ as F and G are sheaves.

Full: Let F and G be in $\mathrm{LCSh}(X)$ and let $\varphi\colon\mu(F)\to\mu(G)$ be a morphism of representations. For every $x \in X$, the constant path γ_x yields a morphism $f_x \to F_x \to G_x$. If U_x is an open neighbourhood of x where both F and G are constant, then we get a morphism $f(U_x)$ rendering the diagram

$$\begin{tikzcd} F(U_x) \arrow[r, dotted, "f(U_x)"] \arrow[d, "\cong"'] & G(U_x) \arrow[d, "\cong"] \\ F_x \arrow[r, "f_x"] & G_x \end{tikzcd}$$

commutative. Since X is assumed to be locally path-connected, we may assume that U_x is path-connected. For every $y \in U_x$ we choose a path δ_y from x to y. Then the diagrams

$$\begin{tikzcd} F_x \arrow[r, "f(U_x)_x"] \arrow[d, "\mu(F,\delta_y)"', "\cong"] & G_x \arrow[d, "\mu(G,\delta_y)", "\cong"'] \\ F_y \arrow[r, "f(U_x)_y"'] & G_y \end{tikzcd} \qquad \begin{tikzcd} F_x \arrow[r, "f_x"] \arrow[d, "\mu(F,\delta_y)"', "\cong"] & G_x \arrow[d, "\mu(G,\delta_y)", "\cong"'] \\ F_y \arrow[r, "f_y"'] & G_y \end{tikzcd}$$

both commute and $f_x = f(U_x)_x$, hence $f_y = f(U_x)$. Thus the morphisms $(f(U_x))_{x\in X}$ glue to a morphism $f\colon F \to G$ such that $\mu(f) = \varphi$. □

Corollary 20.9. *If X is path-connected, simply connected, and locally path-connected, the monodromy functor*

$$\mu\colon \mathrm{LCSh}(X) \longrightarrow \mathrm{Rep}(\Pi_1(X), \mathrm{Set})$$

is an equivalence and every locally constant sheaf on X is constant.

Proof. Since X is path-connnected and simply connected, the canonical functor $p\colon \Pi_1(X) \to \{*\}$ to the trivial category with one object is an equivalence (Lemma 18.5 (ii)). A diagram chase in the commutative diagram

$$\begin{tikzcd}
\mathrm{CSh}(X) \arrow[r, "\mu_0", "\cong"'] \arrow[d, hook] & \mathrm{Rep}_0(\Pi_1(X), \mathrm{Set}) \arrow[d, hook] & \mathrm{Rep}_0(\{*\}, \mathrm{Set}) \arrow[l, "p^*"', "\cong"] \arrow[d, "\cong"] \\
\mathrm{LCSh}(X) \arrow[r, hook, "\mu"] & \mathrm{Rep}(\Pi_1(X), \mathrm{Set}) & \mathrm{Rep}(\{*\}, \mathrm{Set}) \arrow[l, "p^*"', "\cong"]
\end{tikzcd}$$

shows that the monodromy functor μ is essentially surjective. Using that the μ is fully faithful, we see that it is an equivalence of categories. Thus also the inclusion $\mathrm{CSh}(X) \hookrightarrow \mathrm{LCSh}(X)$ an equivalence. □

Theorem 20.10. *Assume that X is*

- *locally path-connected and*
- *admits an open cover of path-connected and simply conected open subsets which is stable by finite intersections.*

Then the monodromy functor

$$\mu\colon \mathrm{LCSh}(X) \longrightarrow \mathrm{Rep}(\Pi_1(X), \mathrm{Set})$$

is an equivalence of categories.

Proof. We only have to show that μ is essentially surjective (since it is fully faithful according to Theorem 20.8). Let $(U_i)_{i\in I}$ be an open cover of path-connected and simply conected open subsets which is stable by finite intersections. The inclusion maps $U_i \hookrightarrow X$ induce functors $\Pi_1(U_i) \to \Pi_1(X)$ which induce functors

$$\mathrm{Rep}(\Pi_1(X), \mathrm{Set}) \longrightarrow \mathrm{Rep}(\Pi_1(U_i), \mathrm{Set}), \quad \varrho \mapsto \varrho|_{U_i}$$

Let $\varrho \in \mathrm{Rep}(\Pi_1(X), \mathrm{Set})$. Since U_i is path-connected, simply connected, and locally path-connected for every $i \in I$, the representations $(\varrho|_{U_i})_{i\in I}$ correspond to unique locally constant sheves $(F_i \in \mathrm{LCSh}(U_i))_{i\in I}$ by Corollary 20.9. Since the $(U_i)_{i\in I}$ are stable by intersection and since the $(F_i)_{i\in I}$ are compatible by design, we can glue and get a locally constant sheaf F such that $F|_{U_i} = F_i$.

It remains to show that there exists a natural isomorphism $\mu(F) \cong \varrho$. Since $\mu(F_i) \cong \varrho|_{U_i}$ we get induced isomorphisms

$$\varphi_x\colon F_x \xrightarrow{\cong} \varrho(x)$$

for every $x \in X$. We have to show that $(\varphi_x)_{x\in X}$ is a natural transformation, i.e. that for every $x, y \in X$ and every path γ from x to y the diagram

$$(\clubsuit) \qquad \begin{tikzcd}
F_x \arrow[r, "\mu(F,\gamma)"] \arrow[d, "\varphi_x"'] & F_y \arrow[d, "\varphi_y"] \\
\varrho(x) \arrow[r, "\varrho(\gamma)"'] & \varrho(y)
\end{tikzcd}$$

is commutative.[28] So let $\gamma:[0,1]\to X$ be a path from x to y. We can decompose

$$\gamma=\gamma_1\cdot\ldots\cdot\gamma_n$$

into smaller paths $(\gamma_k)_k$ such that there exist for every $k\in\{1,\ldots,n\}$ and $i_k\in I$ such that $\mathrm{im}(\gamma_k)\subset U_{i_k}$. Then every square in the commutative diagram

$$\begin{tikzcd} F_{\gamma_1(0)} \arrow[r, "\mu(F,\gamma_1)"] \arrow[d, "\varphi_{\gamma_1(0)}"'] & F_{\gamma_1(1)} \arrow[r, "\mu(F,\gamma_2)"] \arrow[d, "\varphi_{\gamma_1(1)}"] & {} & \cdots\cdots & {} \arrow[r, "\mu(F,\gamma_{n-1})"] & F_{\gamma_n(0)} \arrow[r, "\mu(F,\gamma_n)"] \arrow[d, "\varphi_{\gamma_n(0)}"'] & F_{\gamma_n(1)} \arrow[d, "\varphi_{\gamma_n(1)}"] \\ \varrho(\gamma_1(0)) \arrow[r, "\varrho(\gamma_1)"] & \varrho(\gamma_1(1)) \arrow[r, "\varrho(\gamma_2)"] & {} & \cdots\cdots & {} \arrow[r, "\varrho(\gamma_{n-1})"] & \varrho(\gamma_n(0)) \arrow[r, "\varrho(\gamma_n)"] & \varrho(\gamma_n(1)) \end{tikzcd}$$

commutes by design, hence the square (♣) commutes by functoriality. □

Corollary 20.11. *Assume that X is*

- *path-connected,*
- *locally path-connected, and*
- *admits an open cover of path-connected and simply conected open subsets which is stable by finite intersections.*

Then for every $x\in X$ there is an equivalence

$$\mathrm{LCSh}(X)\xrightarrow{\simeq}\mathrm{Rep}(\pi_1(X,x),\mathrm{Set}).$$

Proof. This is immediate from Lemma 18.5 (i) and Theorem 20.10. □

Corollary 20.12. *Assume that X is*

- *path-connected,*
- *locally path-connected, and*
- *admits an open cover of path-connected and simply conected open subsets which is stable by finite intersections.*

The X is simly connected if and only if every locally constant sheaf on X is constant.

Proof. Since X is path-connected, for any $x\in X$ the canonical functor $x:\pi_1(X,x)\to\Pi_1(X)$ is an equivalence of categories. By Proposition 20.7 and Theorem 20.10 the functor μ_0 and μ both are equivalences. Hence we get a commutative diagram

$$\begin{tikzcd} \mathrm{CSh}(X) \arrow[r, "\mu_0", "\cong"'] \arrow[d, hook] & \mathrm{Rep}_0(\Pi_1(X),\mathrm{Set}) \arrow[d, hook] & \mathrm{Rep}_0(\pi_1(X,x),\mathrm{Set}) \arrow[l, "x^*"', "\cong"] \arrow[d, hook] \\ \mathrm{LCSh}(X) \arrow[r, "\mu", "\cong"'] & \mathrm{Rep}(\Pi_1(X),\mathrm{Set}) & \mathrm{Rep}(\pi_1(X,x),\mathrm{Set}) \arrow[l, "x^*"', "\cong"] \end{tikzcd}$$

wherein the one vertical morphism is an equivalence if and only iff all vertical morphism. This implies the claim. □

Lemma 20.13. *Let $p:E\to X$ be a fibre bundle. Then for every path $\gamma:[0,1]\to X$ and every $e\in p^{-1}(\{\gamma(0)\})$ there exists a **lifted path** $\lambda(\gamma,e):[0,1]\to E$, i.e. $p\circ\lambda(\gamma,e)=\gamma$. If p is a covering, then the lifted path $\lambda(\gamma,e)$ is uniquely determinded by these conditions.*

Proof. Exercise. □

[28]Note that a natural transformation is a natural isomorphism if and only if it maps every object to an isomorphism.

Exercise 20.14. Find a local homeomorphism $p: E \to X$ and a path $\gamma: [0,1] \to X$ and a point $e_0 \in p^{-1}(\{\gamma(0)\})$ such that there does not exists a lifted path $\lambda(\gamma, e_0): [0,1] \to E$ as in Lemma 20.13.

Lemma 20.15. *Assume that X is*

- *locally path-connected and*
- *admits an open cover of path-connected and simply conected open subsets which is stable by finite intersections.*

Let $p: E \to X$ be a covering.

(i) *There exists a representation*
$$\begin{aligned} \varrho(p) : \Pi_1(X) &\longrightarrow \mathrm{Set}, \\ X \ni x &\mapsto E_x, \\ [\gamma: [0,1] \to X] &\mapsto \big(E_{\gamma(0)} \to E_{\gamma(1)}, e \mapsto \lambda(\gamma, e)(1)\big), \end{aligned}$$
where $\lambda(\gamma, e_{\gamma(0)})$ is the unique lift of γ to E which starts at $e_{\gamma(0)}$ (Lemma 20.13).

(ii) *For every $x \in X$ and every $e \in E_x$ we get an induced group action*
$$\varrho(p,x) : \pi_1(X,x) \longrightarrow \mathrm{Aut}_{\mathrm{Set}}(E_x), \quad [\gamma] \mapsto (e \mapsto \lambda(\gamma, e)(1))$$

(iii) *If X is path-connected, then we get for every $x \in X$ and every $e \in E_x$ an induced bijection*
$$\pi_0(E) \overset{\cong}{\longleftrightarrow} \{\pi_1(X,x)\text{–orbits of } E_x\}.$$
In particular, E is path-connected if and only if the action $\varrho(p,x)$ is transitive.

Proof. (i) follows from Lemma 20.13 and (ii) follows immediately from (i). For (iii) we choose for every $y \in X$ a path δ_y from x to y. By Lemma 20.13 we get for every path $\gamma: [0,1] \to E$ a unique path
$$\tilde{\gamma}: [0,1] \to E \text{ such that } \begin{cases} p_F(\tilde{\gamma}) = \delta_{p(\gamma(0))} \cdot p(\gamma) \cdot \delta^{-1}_{p(\gamma(1))} \\ \tilde{\gamma}(0) = e \end{cases}$$
By design we have that $\tilde{\gamma}(1) \in E_x$. Check that we get a well-defined map
$$\pi_0(\acute{\mathrm{E}}(F)) \longrightarrow \{\pi_1(X,x)\text{–orbits of } E_x\}, \quad [\gamma] \mapsto \tilde{\gamma}(1)$$
which is a bijection whose inverse map is induced by the inclusion map $E_x \hookrightarrow E$. ☐

Let us summarise the equivalence we have proven:

Theorem 20.16. *Assume that X is*

- *locally path-connected and*
- *admits an open cover of path-connected and simply conected open subsets which is stable by finite intersections.*

Then we have equivalences of categories:

$$\begin{tikzcd}
\mathrm{LCSh}(X) \arrow[rr, bend left, "\acute{\mathrm{E}}"] \arrow[dr, bend left, "\mu"] & & \mathrm{Cov}(X) \arrow[ll, "\Gamma"] \arrow[dl, bend left, "\varrho"] \\
& \mathrm{Rep}(\Pi_1(X), \mathrm{Set}) \arrow[ul, bend left, "\alpha"] \arrow[ur, bend left, "\beta"] &
\end{tikzcd}$$

where

- *$\acute{\mathrm{E}}$ is the étale space functor (Definition 16.8),*
- *Γ is the sheaf of sections functor (Definition 16.3),*

- *μ is the monodromy functor (Definition 20.6),*
- *ϱ sends a covering $p: E \to X$ to the representation $\varrho_p: \Pi_1(X) \to \mathrm{Set}$ (Lemma 20.13),*
- *α sends a representation $\varrho: \Pi_1(X) \to \mathrm{Set}$ to the sheaf which is glued together form constant sheaves $F_{U_x} \cong \varrho(x)$ for path-connected and simply-connected neighbourhoods U_x for every $x \in X$ (Proof of Theorem 20.10), and*
- *β sends a representation $\varrho: \Pi_1(X) \to \mathrm{Set}$ to the covering space which is glued from the constant coverings $U_x \times \varrho(x) \to U_x$ for path-connected and simply-connected neighbourhoods U_x for every $x \in X$.*

21. Examples and applications

Lemma 21.1. *Every topological manifold is locally path-connected and admits an open cover of path-connected and simply conected open subsets which is stable by finite intersections. A manifold is connected if and only if it is path-connected.*

Proof. Exercise. ☐

Example 21.2. We want to describe locally constant sheaves on the n-sphere

$$S^n := \{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\}$$

for every $n \in \mathbf{N}$ where $\|-\|$ denotes the euclidean norm on $\mathbf{R}^{n+1}$.

The 0-sphere is a two-point space with the discrete topology, hence $\mathrm{LCSh}(S^0) \cong \mathrm{Set} \times \mathrm{Set}$.

For $n \geq 2$ and any $x \in S^n$ we have that $\pi_1(S^n, x) \cong \{1\}$, hence $\mathrm{CSh}(S^n) \cong \mathrm{LCSh}(S^n) \cong \mathrm{Set}$.

For the 1-sphere let us identify

$$S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$$

The fundamental group $\pi_1(S^1, 1)$ is isomorphic to $\mathbf{Z}$ and via the morphism which sends the homotopy class of the loop

$$\gamma_1 : [0,1] \to S^1, \quad t \mapsto \exp(2\pi \mathbf{i} t)$$

By Corollary 20.11 we have an equivalence of categories

$$\mathrm{LCSh}(S^1) \xrightarrow{\simeq} \mathrm{Rep}(\mathbf{Z}, \mathrm{Set}).$$

A $\mathbf{Z}$-representation ϱ in Set is a set $M := \varrho(\{*\})$ together with a bijection $\sigma := \varrho(1): M \xrightarrow{\cong} M$. Then we have

$$\varrho(k) = \begin{cases} \sigma^k & (k \geq 1), \\ \mathrm{id}_m & (k = 0), \\ (\sigma^{-1})^{-k} & (k \leq -1) \end{cases}$$

By Theorem 20.16 we have an equivalence

$$\varrho : \mathrm{Cov}(S^1) \longrightarrow \mathrm{Rep}(\mathbf{Z}, \mathrm{Set})$$

which sends a covering $p: E \to S^1$ the the representation

$$\begin{aligned} \varrho(p) : \mathbf{Z} &\longrightarrow \mathrm{Set}, \\ * &\mapsto E_1 = p^{-1}(\{1\}), \\ k &\mapsto (e \mapsto \lambda(\gamma_1^k, e)(1), \end{aligned}$$

where $\lambda(\gamma_1^k, e)$ is the unique lift of the k-fold concatenation γ_1^k of γ_1 (Lemma 20.15). By the very same lemma, the space E is path-connected if and only if the representation $\varrho(p)$ acts transitively on M, i.e.

$$M = \{\varrho(k)(m) \mid k \in \mathbf{Z}\}$$

for every $m \in M$. Hence M has to be a countable set if E is path-connected. Fixing an element $m \in M$ we get a surjective map of $\mathbf{Z}$-representations

$$\varphi : \mathbf{Z} \longrightarrow M, k \mapsto \varrho(k)(m).$$

Now we have two possible cases:

(1) The map φ is injective, hence a bijection. Thus φ is an isomorhism of representations.
(2) There exist a minimal positive integer $n \in \mathbf{Z}_{\geq 1}$ such that $\varphi(n) = m$. In this case φ induces a bijection $\overline{\varphi} : \mathbf{Z}/n \to M$ which is an isomorphism of representations.

Check that the covering

$$\tilde{p} : \mathbf{R} \longrightarrow S^1, \quad t \mapsto \exp(2x\mathbf{i}t)$$

corresponds to the $\mathbf{Z}$-representation of case (1) and that, for $n \in \mathbf{Z}_{\geq 1}$, the covering

$$p_n : S^1 \longrightarrow S^1, z \mapsto z^n$$

correponds to the $\mathbf{Z}$-representation of case (2).

Thus the locally constant sheaves F whose étalé space $\acute{\mathrm{E}}(F)$ is path-connected are isomorphic to the sheaves of sections $\Gamma(\mathbf{R}, \tilde{p})$ or $\Gamma(S^1, p_n)$ for $n \in \mathbf{Z}_{\geq 1}$. For an arbitrary locally constant sheaf F its étalé space $\acute{\mathrm{E}}(F)$ is a disjoint union of these connected covering spaces.

Example 21.3. For $X = S^1$, the constant sheaf $\underline{R}$ on S^1 has étalé space $\acute{\mathrm{E}}(\underline{R})$ which is in bijection to the infinite cyclinder $S^1 \times \mathbf{R}$, but its topology is different since each fibre $\acute{\mathrm{E}}(\underline{R})_x$ is the real numbers equipped with the discrete topology.

Example 21.4. We want to describe locally constant sheaves on the torus $T := S^1 \times S^1$. For every $x \in T$ we have $\pi_1(T, x) \cong \mathbf{Z} \times \mathbf{Z}$. Hence

$$\mathrm{LCSh}(T) \xrightarrow{\simeq} \mathrm{Rep}(\mathbf{Z} \times \mathbf{Z}, \mathrm{Set})$$

and a $\mathbf{Z} \times \mathbf{Z}$-representation ϱ in Set is a set $M = \varrho(\{*\})$ together with two commuting bijections $\sigma, \tau : M \xrightarrow{\cong} M$ i.e. $\tau \circ \sigma = \sigma \circ \tau$. Then one can see that the connected étalé spaces $\acute{\mathrm{E}}(F)$ correspond to

$$\mathbf{R}^2 \longrightarrow T, \mapsto (s,t) \mapsto (\exp(2\pi\mathbf{i}s), \exp(2\pi\mathbf{i}t))$$

or to

$$S^1 \times \mathbf{R} \longrightarrow T, (x,t) \mapsto (x^n, \exp(2\pi\mathbf{i}t)) \text{ resp. } (\exp(2\pi\mathbf{i}t), x^n)$$

or to

$$T \longrightarrow T, (x,y) \mapsto (x^n, y^m)$$

for $n, m \in \mathbf{N}_{\geq 1}$.

Recall from last lecture. Let X be a topological space.

- We have equivalences of categories

$$\begin{tikzcd} \mathrm{LCSh}(X) \arrow[r, bend left, "\acute{\mathrm{E}}"] & \mathrm{Cov}(X) \arrow[l, bend left, "\Gamma"] \end{tikzcd}$$

- If X is locally path-connected and admits an open cover of path-connected and simply connected open subsets which is stable by finite intersections, then we have equivalences

$$\begin{tikzcd} \mathrm{LCSh}(X) \arrow[rr, bend left, "\text{É}"] \arrow[dr, "\mu"'] & & \mathrm{Cov}(X) \arrow[ll, bend left, "\Gamma"] \arrow[dl, "\varrho"] \\ & \mathrm{Rep}(\Pi_1(X),\mathrm{Set}) \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

- The functor ϱ is defined in terms of the following path-lifting property: Let $p\colon E \to X$ be a fibre bundle. Then for every path $\gamma\colon[0,1] \to X$ and every $e \in p^{-1}(\{\gamma(0)\})$ there exists a **lifted path** $\lambda(\gamma,e)\colon[0,1] \to E$, i.e. $p \circ \lambda(\gamma,e) = \gamma$. If p is a covering, then the lifted path $\lambda(\gamma,e)$ is uniquely determinded by these conditions.

We now can prove homotopy invariance of sheaf cohomology in a different way.

Theorem 21.5. *Let $f\colon X \to Y$ be homotopy equivalence. Then for every locally constant sheaf G on Y we have an isomorphism*

$$\mathrm{H}^*(Y,G) \xrightarrow{\cong} \mathrm{H}^*(X,f^{-1}G).$$

Here we assume that the results about locally constant sheaves and covering spaces have variants for sheaves of R-modules for a ring R.

Proof. For a covering $p\colon E \to Y$ the pullback $f^*(p)\colon f^*E := E \times_Y X \to X$ is a covering and we get a functor $f^*\colon \mathrm{Cov}(Y) \to \mathrm{Cov}(X)$. Check that the diagram

$$\begin{tikzcd} \mathrm{LCSh}(Y) \arrow[r, "\text{É}", "\cong"'] \arrow[d, "f^{-1}"'] & \mathrm{Cov}(Y) \arrow[d, "f^*"] \\ \mathrm{LCSh}(X) \arrow[r, "\text{É}", "\cong"'] & \mathrm{Cov}(X) \end{tikzcd}$$

is commutative. Now let $g\colon Y \to X$ be a homotopy inverse of f. Check that the functors

$$f^*\colon \mathrm{Cov}(Y) \rightleftarrows \mathrm{Cov}(X) : g^*$$

are inverse equivalences of categories. Hence the functors

$$f^{-1}\colon \mathrm{LCSh}(Y) \rightleftarrows \mathrm{LCSh}(X) : g^{-1}$$

are inverse equivalences of categories. For every locally constant sheaf G on Y we have

$$(\clubsuit) \qquad \begin{aligned} \Gamma(Y,G) &\cong \mathrm{Hom}_{\mathrm{LCSh}(Y)}(\underline{R}_Y,G) \\ &\cong \mathrm{Hom}_{\mathrm{LCSh}(Y)}(\underline{R}_Y,g^{-1}f^{-1}G) \\ &\cong \mathrm{Hom}_{\mathrm{LCSh}(X)}(f^{-1}\underline{R}_Y,f^{-1}G) \\ &\cong \mathrm{Hom}_{\mathrm{LCSh}(X)}(\underline{R}_X,f^{-1}G) \\ &\cong \Gamma(X,f^{-1}G). \end{aligned}$$

Check that for every locally constant sheaf G on Y, the Godement sheaf $\mathcal{C}(G)$ (Definition 9.16) is locally constant as well. Hence the Godement resolution

$$0 \longrightarrow G \xrightarrow{\varphi} \mathcal{C}^0(G) \xrightarrow{d^0} \mathcal{C}^1(G) \xrightarrow{d^1} \mathcal{C}^2(G) \xrightarrow{d^2} \dots$$

from Construction 9.19 is a flasque resolution of locally constant sheaves. Hence $\mathrm{H}^*(Y,G)$ can be equally computed in terms of the right-derived functors of the restricted global sections functor $\Gamma(Y,-)\colon \mathrm{LCSh}(Y) \to \mathrm{Set}$. Check that $f^{-1}\mathcal{C}^n(G) \cong \mathcal{C}^n(f^{-1}G)$ for every $n \in \mathbf{N}$ which yields together with ($\clubsuit$) the desired assertion. $\square$

22. Outlook

Here are some suggestions of things you could study with the theory from this course (depending on your background).

22.1. **Sheaves on categories with topology.** So far we have studied presheaves and sheaves on topological spaces. This can be generalised to a more general setting which has many interesting applications (e.g. in algebraic geometry). Given a topological space X, a presheaf F on X is a functor

$$F : \mathrm{Open}(X) \longrightarrow \mathrm{Set}$$

This motivates us to the following.

Definition 22.1. Let $\mathcal{C}$ be a category. A **presheaf F on $\mathcal{C}$** is a functor $F{:}\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. The **category of presheaves on $\mathcal{C}$** is the functor category

$$\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}),$$

i.e. morphisms of presheaves are natural transformations.

In order to make sense out of the sheaf condition we need more structure on a category.

Definition 22.2. Let $\mathcal{C}$ be a category. A **pretopology** τ on $\mathcal{C}$ is given by a set $\mathrm{Cov}_\tau(\mathcal{C})$ of **τ-covers**, i.e. a set of families of morphism with fixed target $(U_i \to U)_i$ satisfying the following conditions:

(i) Every isomorphism $V \to U$ is a cover.
(ii) For a cover $(U_i \to U)_i$ and covers $(U_{ij} \to U_i)_j$ for every i, the induced family $(U_{ij} \to U_i \to U)_{i,j}$ is a cover.
(iii) For a cover $(U_i \to U)_i$ and any morphism $V \to U$ the fibre products $U_i \times_U V$ exist and $(U_i \times_U V \to V)_i$ is a cover.

We say that a presheaf $F{:}\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is a **τ-sheaf** iff for every τ-cover $(f_i{:}U_i \to U)_i$ the induced diagram

$$F(U) \xrightarrow{F(f)} \prod_i F(U_i) \underset{F(\mathrm{pr}_2)}{\overset{F(\mathrm{pr}_1)}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

is an equaliser diagram.

Example 22.3. Let X be a topological space. We equip the category $\mathrm{Open}(X)$ with the pretopology whose covers are precisely those families $(U_i \to U)_i$ which satisfy $U = \bigcup_i U_i$. Then a sheaf on $\mathrm{Open}(X)$ with respect to this topology is precisely a sheaf on X in the usual sense.

In algebraic geometry one studies zero sets of polynomials which are equipped with the Zariski topology. These assemble to schemes, that is a certain type of locally ringed spaces $(X, \mathcal{O}_X)$ that "locally look like rings". The topology on a scheme X, the so-called **Zariski topology** is a very coarse topology and does not yield a cohmology theory whith sufficient properties. In order to overcome this, one studies the **étale topology** on schemes which is finer than the Zariksi topology (i.e. every Zariski-open subscheme is also open in the with respect to the étale topology). With the étale topology one can study étale sheaves (in the sense of Definition 22.2) and, by taking the right-derived functors of the left-exact global sections functor, one gets **étale cohomology**. For further you may consider:

- Milne, J.S.: *Lectures on Ètale Cohomology*, Version 2.21, March 22, 2013. `https://www.jmilne.org/math/CourseNotes/LEC.pdf`
- Fu, Lei: *Étale Cohomology Theory*, Nankai Tracts in Mathematics – Vol. 13, 2011.

22.2. **Direct image with compact support and Verdier duality.**

Recall 22.4. Let R be a ring. For a sheaf of R-modules F on a topological space X there is an adjunction

$$F \otimes_R (-) \colon \mathrm{Sh}(X,R) \leftrightarrows \mathrm{Sh}(X,R) \colon \underline{\mathrm{Hom}}_{\mathrm{Sh}(X,R)}(F,-).$$

For every map of topological spaces $f\colon X \to Y$ there is an adjunction

$$f^{-1} \colon \mathrm{Sh}(Y,R) \leftrightarrows \mathrm{Sh}(X,R) \colon f_*.$$

If $j\colon U \hookrightarrow X$ is the inclusion of an open subset, then there is an adjunction

$$j_! \colon \mathrm{Sh}(U) \leftrightarrows \mathrm{Sh}(X) \colon f^{-1}$$

where for a sheaf F on U the sheaf $j_!F$ is the extension by zero.

Definition 22.5. Let $f\colon X \to Y$ be a continuous map between topological spaces. We define the **direct image with compact support** to be the functor

$$f_! \colon \mathrm{Sh}(X,R) \longrightarrow \mathrm{Sh}(Y,R), \quad F \mapsto f_!F$$

by setting for an open subset V of Y

$$f_!F(V) := \left\{ s \in f_*F(V) \mid \mathrm{supp}(s) \to V \text{ is proper} \right\}$$

where $\mathrm{supp}(s) = \{x \in X \mid s_x \neq 0 \text{ in } F_x\}$ is the **support** of the section s.

Lemma 22.6. *Let $f\colon X \to Y$ be a continuous map.*

(i) If f is proper, then $f_! = f_*$ as functors.
(ii) If f is an open immersrion, then $f_!$ is the extension by zero.

Proof. Exercise. □

Definition 22.7. Let $\mathcal{A}$ be an abelian category. We define its **derived category** $\mathrm{D}(\mathcal{A})$ to be the category with the same objects as $\mathrm{Ch}^\bullet(\mathcal{A})$ and whose morphism between two compelexes $A^\bullet$ and $B^\bullet$ are defined to be

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet,B^\bullet) := \mathrm{Hom}_{\mathrm{Ch}^\bullet(\mathcal{A})}(A^\bullet,B^\bullet)/\{\mathrm{H}^* \text{-isomorphisms}\},$$

i.e. morphisms of chain complexes up to morphisms which are isomorphisms on cohomology.[29] We denote by $\mathrm{D}^b(\mathcal{A})$ the full subcategory of $\mathrm{D}(\mathcal{A})$ which is spanned by bounded complexes.

Theorem 22.8 (Verider duality)**.** *There exists a functor*

$$f^! \colon \mathrm{D}^b(\mathrm{Sh}(Y,R)) \longrightarrow \mathrm{D}^b(\mathrm{Sh}(X,R))$$

such that we get an adjunction

$$\mathrm{D}^b(f_!) \colon \mathrm{D}^b(\mathrm{Sh}(X,R)) \leftrightarrows \mathrm{D}^b(\mathrm{Sh}(Y,R)) \colon f^!.$$

A proof and more background can be found in the following books:

- S.I. Gelfand, Yu. Manin: *Methods of homological algebra* (second edition), Springer, 2003.
- Dimca, Alexander: *Sheaves in Topology*, Springer Universitext, 2004.

[29] We omit here some technical issues to be cleared in order that the definition makes sense.

22.3. **de Rham cohomology.** Let M be a (real) smooth manifold. For $n \in \mathbf{N}$ there is the **sheaf of n-forms** Ω^n_M which send an open subset U to the $\mathbf{R}$-vector space of n-forms on U. There are differential morphism and we get the **de Rham complex**

$$0 \longrightarrow \underline{\mathbf{R}} \longrightarrow \Omega^1_M \longrightarrow \Omega^2_M \longrightarrow \dots$$

Definition 22.9. For $n \in \mathbf{N}$ we define its **n-th de Rham cohomology** to be

$$\mathrm{H}^n_{\mathrm{dR}}(M,\mathbf{R}) := \mathrm{H}^n(\Omega^\bullet_M(M)).$$

For every $x \in M$ the complex on stalks of the de Rham complex is exact by the Poincarè Lemma so that the **de Rham complex** is exact. One can show that the sheaves Ω^n_M are **soft sheaves** and that soft sheaves are $\Gamma(M,-)$-acyclic. This implies the following.

Theorem 22.10. *For every manifold M and every $n \in \mathbf{N}$ there is an isomorphism*

$$\mathrm{H}_{\mathrm{dR}}(M,\mathbf{R}) \cong \mathrm{H}^n(M,\underline{\mathbf{R}}).$$

22.4. **Riemann-Hilbert correspondence.** For a complex manifold M, there is an equivalence between the the following categories:

(i) Locally constant sheaves of finitely dimensional $\mathbf{C}$-vector spaces on M.
(ii) Holomorphic vector bundles on M together with an integrable connection.

For a first look, one could have a look at:

- Conrad, Brian: *Classical Motivation for the Riemann-Hilbert correspondence*, lecture notes.
  `http://math.stanford.edu/~conrad/papers/rhtalk.pdf`
- Dimca, Alexander: *Sheaves in Topology*, Springer Universitext, 2004.

Notation

Here is some notation used in this lecture notes. If there are some symbols where you do not know what they mean, please write me an email.

Symbol	**Description**	**Reference**
Ab	the category of abelian groups	
$\mathrm{Bun}(X)$	the category of fibre bundles over a space X	17.3
$\mathrm{Ch}^\bullet(\mathcal{A})$	(cochain) complexes in an abelian category $\mathcal{A}$	4.12 (iii)
$\mathrm{Cov}(X)$	the category of coverings of a space X	17.3
$\mathrm{É}(F)$	étalé space of a presheaf F	16.8
$\Gamma(E,p)$	sheaf of sections of a map $p{:}E \to X$	16.3
$\mathrm{H}^n(A^\bullet)$	n-th cohomology of a complex $A^\bullet$	4.12 (iv)
$\mathrm{H}^n(X,F)$	n-th sheaf cohomology of a sheaf F on a space X	9.2
$\check{\mathrm{H}}^n(\mathcal{U},F)$	n-th Čech cohomology of a sheaf F for a cover $\mathcal{U}$	10.2
$\check{\mathrm{H}}^n(X,F)$	n-th absolute Čech cohomology of a sheaf F on a space X	11.4
iff	"if and only if" (abbreviation only used in definitions)	
$\mathrm{LH}(X)$	the category of local homeomorphisms over a space X	16.5
π_X	the unique map $\pi_X{:}X \to \{*\}$ for a space X	
$\underline{R}$	constant sheaf with value R	
$\mathrm{R}^n F$	the n-th right derived functor for a left-exact functor F	4.10
$\mathrm{Sh}(X)$	sheaves of sets on a topological space X	
$\mathrm{Sh}(X,R)$	sheaves of R-modules on a topological space X for a ring R	
$\mathrm{Top}_{/X}$	topological spaces over a space X	16.1

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`https://webusers.imj-prg.fr/~pierre.schapira/lectnotes/AlTo.pdf`

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