# K-theory of the integers and the Kummer-Vandiver Conjecture

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Though algebraic K-theory is a vital research area, the algebraic K-theory of the integers $\mathrm{K}_*(\mathbf{Z})$ is still not fully known. Letting $\mathrm{B}_k$ be the $k$-th Bernoulli number and setting $\frac{\mathrm{B}_k}{4k} = \frac{c_k}{w_{2k}}$ with $(c_k, w_{2k}) = 1$ we know the following:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\mathrm{K}_{8k+n}(\mathbf{Z})$ | ? | $\mathbf{Z} \oplus \mathbf{Z}/2$ | $\# = 2c_{2k+1}$ | $\mathbf{Z}/2w_{4k+2}$ | ? | $\mathbf{Z}$ | $\# = c_{2k+2}$ | $\mathbf{Z}/w_{4k+4}$ |

As a consequence we obtain a link with the Riemann zeta function (for $k \geq 1$), namely

$$\frac{|\mathrm{K}_{4k-2}(\mathbf{Z})|}{|\mathrm{K}_{4k-1}(\mathbf{Z})|} = \frac{\mathrm{B}_k}{4k} = \frac{(-1)^k}{2}\zeta(1-2k).$$

In this seminar we want to better understand the higher algebraic K-theory of the integers as well as its relation to arithmetic. We will study Quillen's classical finiteness result for the K-theory of integers of number fields and connections between K-theory and étale cohomology. The latter will be used to link the conjectured vanishing of the groups $\mathrm{K}_{4n}(\mathbf{Z})$ (for $n \geq 1$) with the Kummer–Vandiver conjecture about cyclotomic extensions of the rational numbers [Kur92]. In the second half of the seminar we will study the proof of the known case that $\mathrm{K}_4(\mathbf{Z}) = 0$ whose constributions can be summed up as follows:

- Lee-Szczarba showed that $\mathrm{K}_4(\mathbf{Z})$ (and $\mathrm{K}_5(\mathbf{Z})$) contains no $p$-torsion for prime numbers $p > 5$ [LS78]. Soulé extended their arguments and showed that $\mathrm{K}_4(\mathbf{Z})$ (and $\mathrm{K}_5(\mathbf{Z})$) does not contain 5-torsion [Sou78].
- Weibel completely determined the 2-torsion of $\mathrm{K}_*(\mathbf{Z})$ [Wei97]. This resuls depends on the work of Voevodsky [Voe03][1], Suslin-Voevodsky [SV00], and Bloch-Lichtenbaum [BL94] on the Milnor conjecture, the Bloch-Kato conjecture, and the Quillen-Lichtenbaum conjecture. A gap was bridged by Rognes-Weibel [RW00, §5].
- Rognes showed that $\mathrm{K}_4(\mathbf{Z})$ does not have 3-torsion and concluded that $\mathrm{K}_4(\mathbf{Z}) = 0$. This uses previous work of Rognes [Rog92] as well as a homology computation by Soulé [Sou00].

Furthermore, it was shown recently that $\mathrm{K}_8(\mathbf{Z}) = 0$ [DSEVKM19, Kup19], but this will not be subject of the seminar.

**Apéritif:** One might want to read Soulé's overview article "Algebraic K-theory of the integers"
`https://link.springer.com/chapter/10.1007/BFb0088880`

**Time and Place:** Tuesday: 11:15–12:45 in Heidelberg, Mathematikon, SR 8, and online.
Last session on 31st January: Talk 9 (11:15–12:45, SR 8) and Talk 10 (14:15–15:45, SR 12).
Afterwards we will have a cosy walk along Philosophenweg and dinner in Heidelberg Altstadt.

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[1]Depending on the eventually unpublished preprint [Voe94].

## Talks

**Talk 0: Introduction and overview** (18.10. Christian Dahlhausen)
Introduction of the main players in the seminar, some historial background, and an overview of the seminar's content. Assignment of possibly free talks. Introduction of algebraic K-theory of rings: Define the Grothendieck group $\mathrm{K}_0(A)$ of a ring $A$ and compute it for Dedekind domains. Define the first K-group $\mathrm{K}_1(A)$ and compute it for euclidean rings. Define the K-theory space $\mathrm{K}(A) = \mathrm{BGL}(A)^+$ using (without proof) the universal property of the $+$-construction. State that Quillen constructed a K-theory functor $\mathrm{K}^{\mathrm{Q}}$ from the category of exact categories (with exact functors) to the category of topological spaces and that $\mathrm{K}(A) \simeq \mathrm{K}^{\mathrm{Q}}(\mathrm{Proj}^{\mathrm{fg}}(A))$ for the exact category $\mathrm{Proj}^{\mathrm{fg}}(A)$ of finitely generated projective $A$-modules. The foundational paper is Quillen's article [Qui73b] and a good textbook is [Wei13].

**Talk 1: The Solomon-Tits theorem** (08.11. Marlon Kocher)
This talk is the prequel for the subsequent talk about Quillen's finiteness result [Qui73a]. Introduce the building of a vector space and explain the Solomon-Tits theorem that these buildings have the homotopy type of a bouquet of spheres [Qui73a, Thm. 2].

**Talk 2: Finiteness of the K-theory of the integers** (15.11. Katharina Hübner)
Explain Quillen's proof that the K-theory of rings of integers of number fields is finitely generated [Qui73a]. Define the rank filtration on the category of finitely generated projective modules and on the induced Q-construction. Later in the seminar, we will see an analogous filtration on the spectrum level. State the exact sequence relating different levels of this filtration [Thm. 3] and deduce [Thm 1] from it. Afterwards, explain the proof of [Thm. 3] as detailed as time permits.

**Talk 3: K-theory and étale cohomology** (29.11. Tim Holzschuh)
Relate the K-theory of integers of number fields with their étale cohomology via the Chern character [Kur92, §2] so that we are prepared for the subsequent talk. In fact, we do not necessarily need the full result [Prop. 2.1]; it suffices to show that for a number field $F$ with ring of integers $\mathcal{O}_F$ the Chern character

$$\mathrm{ch}\colon\ \mathrm{K}_{2r-2}(\mathcal{O}_F) \longrightarrow \mathrm{H}^2(\mathcal{O}_F[\tfrac{1}{p}], \mathbf{Z}_p(r))$$

is surjective. This is based on Soulé's paper [Sou79] and Dwyer-Friedlander's paper [DF85].

**Talk 4: K-theory of the integers and cyclotomic fields** (06.12. Max Witzelsperger)
This talk covers Kurihara's article relating K-theory with cyclotomic field extensions $\mathbf{Q}(\mu_p)|\mathbf{Q}$ for prime numbers $p \neq 2$ [Kur92, §1, §3]. Introduce the conjectures of Kummer-Vandiver (and Iwasawa) about the vanishing (resp. cyclicity) of certain eigenspaces of $p$-Sylow subgroups of class groups of cyclotomic extensions of the rational numbers [§0] and relate them to the vanishing (resp. cyclicity) of $\mathrm{H}^2_{\mathrm{et}}(\mathbf{Z}[\frac{1}{p}], \mathbf{Z}(r))$ [Cor. 1.5]. Show that the conjectured vanishing of $\mathrm{K}_{4n}(\mathbf{Z})$ for $n \geq 1$ implies the Kummer-Vandiver conjecture [Prop. 3.7], see also [Wei13, VI.10.9].

**Talk 5: The $p$-torsion of $\mathrm{K}_*(\mathbf{Z})$ for $p > 5$** (13.12. Amine Koubaa)
Sketch the proof that $\mathrm{K}_4(\mathbf{Z})$ does not contain any $p$-torsion for $p \geq 5$ which was shown by Lee-Szczarba [LS78] ($p > 5$) and Soulé [Sou78] ($p = 5$).

**Talk 6: A homology computation** (10.01. Alexander Schmidt)
Explain Soulé's computation that the homology group $\mathrm{H}_1(\mathrm{SL}_4(\mathbf{Z}), \mathrm{St}_4)$ is a finite 2-group [Sou00].

**Talk 7: The 2-torsion of $\mathrm{K}_*(\mathbf{Z})$** (17.01. Rustam Steingart)
Explain Weibel's computation of the K-theory with coefficients in $\mathbf{Z}/2$ [Wei97, Thm. 7]. Here we use without proof Voevodsky's result that $\mathrm{K}^{\mathrm{M}}_*(F)/2 \cong \mathrm{H}^*_{\mathrm{et}}(F; \mathbf{Z}/2)$ for fields $F$ of characteristic $\neq 2$ together with the work of Suslin-Voevodsky [SV00] and Bloch-Lichtenbaum [BL94][2] in order

[2] A better reference for the Bloch-Lichtenbaum spectral sequence might be [Lev06].

to have a spectral sequene

$$E_2^{p,q} = \mathrm{H}_{\mathrm{et}}^{p-q}(F; \mathbf{Z}/2) \Rightarrow \mathrm{K}_{-p-q}(F; \mathbf{Z}/2).$$

Explain the consequences of this spectral sequence for $F = \mathbf{Q}_\ell$ and $F = \mathbf{R}$. Note that the proof for $F = \mathbf{R}$ [Prop. 4] relies on an unjustified assumption, but that this gap is bridged in a subsequent article of Rognes-Weibel [RW00, §5]; this subtlety shall be briefly mentioned and then ignored. As a consequence, deduce the descriptions of $\mathrm{K}_*(\mathbf{Q}; \mathbf{Z}/2)$ and of $\mathrm{K}_*(\mathbf{Z}; \mathbf{Z}/2)$.

**Talk 8: The spectrum level rank filration** (24.01. NN)
This talk covers the results [Rog00, §1–§4] whose proofs rely on [Rog92, §1–9]. First recall some relevant notions from topology which are needed. Introduce the rank filtration $F_\bullet\,\mathrm{K}(R)$ on the algebraic K-theory spectrum $\mathrm{K}(R)$ of a ring $R$ and relate its subquotients $\bar{F}_\bullet\,\mathrm{K}(R)$ with the relative smash product of the universal bundle $\mathrm{EGL}_\bullet(R)$ with the stable building $D(R^\bullet)$ over $\mathrm{GL}_\bullet(R)$ $R$ [Prop. 2.2]. In order to understand better the second component $D(R^\bullet)$ introduce the poset filtration on it [Def. 3.1] and then relate the subquotients of this filtration with the relative smash product of $\mathrm{GL}_\bullet(R)/P_\omega$ with the subquotients on the poset filtration on stable apartments [Prop. 4.2] which will be analysed in the subsequent talk.

**Talk 9: Suspended Tits buildings** (31.01. NN)
This talk covers the results of [Rog00, §5, §6]. Explain the explicit identifications of the poset rank filtration and its subquotients for stable apartments [Prop. 5.1, Prop. 5.4]. Introduce Tits buildings and explain the relation between the Tits buildings of a PID and its fraction field [Lem. 6.1]. Maybe arrange with the subsequent talk's speaker to cover some material from [§7] in order to alleviate their job.

**Talk 10: $\mathrm{K}_4(\mathbf{Z})$ is the trivial group** (31.01. NN)
This talks covers the results of [Rog00, §7, §8]. Introduce the component filtration of stable buldings [Def. 7.1] and explain (as much as time permits) the associated spectral sequence

$$E^1_{s,t} = \mathrm{H}_t(\mathrm{GL}_k(R); Z_s) \Rightarrow \mathrm{H}_{s+t}(\bar{F}_k\,\mathrm{K}(R)).$$

Finally, explain what we can conclude about the rank filtration spectral sequence for $\mathrm{K}(\mathbf{Z})$ modulo the Serre subcategory of finite 2-groups [(8.4)]. Compute the low degrees of the spectrum homology $\mathrm{H}_*(\mathrm{K}(\mathbf{Z}))$ of the spectrum $\mathrm{K}(\mathbf{Z})$ [Thm. 8.5] and subsequently the vanishing of $\mathrm{K}_4(\mathbf{Z})$ [Thm. 8.6].

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