Seminar on Homotopical Category Theory

Wintersemester 2023/24
Dr. Christian Dahlhausen & Lukas Waas

In mathematics, one is mostly not interested in equality of mathematical objects, but in *structural* equality; for instance, all groups with two elements should be considered as equal. In category theory, this idea is formalised by the notion of *isomorphy*. As a matter of fact, many categorical constructions are *universal* and yield objects which are *unique up to unique isomorphism*. However, this is not always the case: Given a left-exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories and assuming that $\mathcal{A}$ has enough injective objects, the right-derived functors $\mathrm{R}^*F\colon \mathcal{A} \to \mathcal{B}$ are defined via *choices* of injective resolutions, since such resolutions are not unique up to unique isomorphism. This construction is ad-hoc and unsatisfactory as one has to check the independece of choices many times. A conceptual solution to this deficiency is to consider the *space* of all injective resolutions: for every injective resolution one takes a point, for every homotopy equivalence between two injective resolutions one attaches a path between the respective points, for every homotopy between homotopy equivalences one attaches a disc, and so on. The resulting space is only unique up to homotopy (a so-called *homotopy type*) and it turns out to be contractible; i.e. injective resolutions are *unique up to contractible choice*. This leads to Homotopical Category Theory and the notion of an ∞-category: with any two objects one associates a *space of morphisms* as opposed to a bare set of morphisms. Ordinary categories (1-categories) are precisely those ∞-categories, up to equivalence, in which every space of morphisms is discrete.

In this seminar we will have a first glimpse into this theory and we want to understand why ∞-categories are useful and how to transfer concepts from ordinary category theory to homotopical category theory. We will not be able to spell out every technical detail, but rather focus on the main concepts and the ideas behind them.

Target audience: Highly motivated and dedicated students who enjoy abstract structures.

Recommended foundations: Basic knowledge about category theory and either topological homotopy theory or homological algebra. We expect every participant to be familar with the content of [Lan21, §1.1] in the beginning of the seminar.

Time and place: Wednesday, 16:15–17:45 Uhr, at Mathematikon.

Literature: We will work through the entire book of Land [Lan21]. Good introductory sources are Chapter 1 of Lurie's book [Lur09] as well as the texts by Groth [Gro20] (arXiv:1007.2925) and Riehl [Rie22] (arXiv:1904.00886). Another good source to keep in mind is Cisinski's book [Cis19].

Registration and contact: If you are interested in participating the seminar, please register at MÜSLI *and* write an email to Christian Dahlhausen (`cdahlhausen@mathi.uni-heidelberg.de`) indicating your relevant background for the seminar. For any inquiries, please contact Christian Dahlhausen or Lukas Waas (`lwaas@mathi.uni-heidelberg.de`).

References

[Cis19] Denis-Charles Cisinski, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, Cambridge, 2019.

[Gro20] Moritz Groth, *A short course on ∞-categories*, Handbook of homotopy theory, CRC Press/Chapman Hall Handb. Math. Ser., CRC Press, Boca Raton, FL, 2020, pp. 549–617.

[Lan21] Markus Land, *Introduction to infinity-categories*, Compact Textbooks in Mathematics, Birkhäuser/Springer, Cham, 2021.

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Rie22] Emily Riehl, *Homotopical categories: from model categories to $(\infty, 1)$-categories*, Stable categories and structured ring spectra, Math. Sci. Res. Inst. Publ., vol. 69, Cambridge Univ. Press, Cambridge, 2022, pp. 5–74.

Guidelines

We are going to cover a lot of material much of which is very intuitive on a conceptual level. However, developing a rigorous machinery in which we express our conceptual ideas can be quite challenging at times. Here are rough instructions on how to draft a talk which makes sure the audience does not drown in a wave of simplex combinatorics.
First things first, you can't simply copy the book word for word, you will certainly run out of time if you try to do so. Thus, your goal should be to figure out what's important and should receive attention, and what can be explained without giving all the details. Here is how to give a nice talk, even when you can't cover every single argument in detail:

- Figure out what the main conceptual ideas and what the techniques you are presenting will be used for. Getting this across to the audience is your main job! Usually, a good talk does not contain more than two or three core ideas! To do this, it is a good idea to have a rough overview on what else is going to happen in future talks and where your results will come in handy in the future.

- Don't get lost in verifying statements from elementary category theory, or from the combinatorics of simplicial sets. Especially with the latter, often the audience may profit a lot more if you illustrate the central idea of proof on some instructive examples, instead of giving a four page inductive proof.

- When you are unsure what is important and what you can safely leave out, we will gladly discuss your draft of the talk with you in person (preferably at least two weeks before the actual talk).

List of talks

Talk 1: Introduction I (18.10.)

Talk 2: Introduction II (25.10.)

Talk 3: From 1-categories to ∞-categories (and back) [Lan21, 1.2.1-1.2.29, 1.2.72-1.2.80] (08.11.)

Talk 4: From simplicial categories to quasi-categories [Lan21, 1.2.30 -1.2.71] (15.11.)

Talk 5: Anodyne maps and fibrations [Lan21, §1.3] (22.11.)

Talk 6: Mapping spaces, joins, and slices [Lan21, §1.4] (29.11.)

Talk 7: Joyal's theorem: on invertible arrows [Lan21, §2.1, §2.2]. (06.12.)

Talk 8: Fully faithfulness, essential surjectivity, and localisations of ∞-categories [Lan21, §2.3, §2.4] (13.12.)

Talk 9: Mapping spaces, fat joins and fat slices [Lan21, §2.5] (20.12.)

Talk 10: (Co)cartesian fibrations [Lan21, §3.1] (10.01.)

Talk 11: Straightening-Unstraightening [Lan21, §3.2, §3.3] (17.01.)

Talk 12: The Yoneda lemma [Lan21, §4.1, §4.2] (24.01.)

Talk 13: Limits and colimits [Lan21, §4.3, §4.4] (31.01.)

Talk 14: Adjoint Functor Theorems [Lan21, §5] (07.02.)